A LONG TIME BEHAVIOR FOR GRADIENT-LIKE SYSTEMS UNDER A WEAK ANGLE CONDITION

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ABSTRACT

We study the long time behavior of solutions of gradient-like systems that admit a strict Lyapunov function. In a recent paper of Merlet and Nguyen [11], the authors showed that if the Lyapunov function satisfies the Kurdyka-Lojasiewicz inequality then the convergence result can be obtained by the angle condition. In this paper, we extend this result by taking a weaker angle condition which was proposed by Huang in [6]. The convergence rates are also obtained under some additional hypotheses.

Keywords: Gradient-like systems, Lyapunov function, Lojasiewicz inequality.

TÓM TẮT

Dáng điệu tiệm cận cho hệ tựa gradient dưới điều kiện góc yếu

Chúng tôi quan tâm dáng điệu tiệm cận của nghiệm hệ tựa gradient, thừa nhận một hàm dạng Lyapunov. Trong một bài báo gần đây của Merlet và Nguyen [11], các tác giả đã chỉ ra rằng nếu hàm Lyapunov thỏa mãn bất đẳng thức Kurdyka-Lojasiewicz thì kết quả hội tụ có thể thu được bởi điều kiện góc. Trong bài báo này, chúng tôi mở rộng kết quả này bằng cách dùng điều kiện góc yếu hơn, được đề ra bởi Huang trong [6]. Tốc độ hội tụ cũng thu được dưới một số điều kiện bổ sung.

Từ khóa: hệ tựa gradient, hàm Lyapunov, bất đẳng thức Lojasiewicz.

1. Introduction

We are interested in the long-time behavior and stability properties of the global solutions of the non-linear differential system,

$$u'(t) = G(u(t)), \qquad t \geq 0, \ u(t) \in \mathbb{R}^n, \tag{1}$$

where $G : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is a continuous vector field on Ω. We assume that the system (1) admits a strict Lyapunov function $F : \mathbb{R}^n \to \mathbb{R}$, that is

$$\frac{d}{dt}\left[F(u)\right](t) \leq 0 \text{ for every solution and every } t \geq 0,$$

and moreover,

$$\frac{d}{dt}\left[F(u)\right](t_0) = 0 \ \Rightarrow \ u(t) = u(t_0) \quad \text{for } t \geq t_0.$$

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The most simple situation is the case of a gradient flow $G = -\nabla F.$ In this case, if F is of class C^1 and bounded from below, we have $F(u(t))$ converges as t tends to infinity.

Moreover, the ω-limit set

$$\omega[u] := \{v \in \Omega \subset \mathbb{R}^n : \exists (t_k) \uparrow \infty \text{ such that } u(t_k) \to v\}$$

is a connected subset of the critical points of F.

We restrict our study to situations for which F satisfies a Lojasiewicz inequality at some point $a \in \omega[u]$, that is: there exists $\theta \in [0,1/2)$, $\gamma > 0$ and $\sigma > 0$ such that

$$|F(v) - F(a)|^{1-\theta} \le \gamma \|\nabla F(v)\|, \quad \forall v \in B(a,\sigma) \cap \Omega,$$

where $B(a,\sigma) = \{v \in \mathbb{R}^n : \|v - a\| < \sigma\}$, and $\|.\|$ denotes the Eclidian norm in $\mathbb{R}^n$. This inequality implies that u has a limit at infinity $u(t) \to a$ as $t \to \infty$, and we even have the stronger result

$$\int_{\mathbb{R}^+} \|u'(t)\| dt < \infty.$$

The importance of the Lojasiewicz inequality comes from a famous result by Lojasiewicz which states that: "if $F: \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is real analytic then such inequality holds in the neighborhood of any point $a \in \Omega$". Certainly, this is non trivial only when a is a critical point of F.

The convergence is not true in general. As a counterexample, Palis and de Melo built a C^∞ "Mexican hat" function (see [12]). They showed that the Euclidean gradient system associated with this energy possesses a global, bounded solution which has the whole unit circle as ω-limit set.

In the papers by Lageman [8,9], by Chill et al. [1,2], by Barta et al. [3,4], by Haraux et al. [5], or the more recent paper by Merlet et al. [11], the authors proved that if F satisfies a Lojasiewicz inequality in a neighborhood of $a \in \omega[u]$ and if $G(u)$ and $-\nabla F(u)$ satisfy an angle condition then $u(t)$ converges to a. We even have convergence rates depending on the Lojasiewicz exponent under an additional comparability condition.

The main goal of this paper is to obtain the convergence results under a more general angle condition which is obtained from the angle condition by adding a negative term. The sequel is organized as follows. In the next section, we set the notations and the main hypotheses. We also give some existing convergence results. In the last section, we prove the convergence result and then we obtain the convergence rates.

2. Some notations and hypotheses

Let us first recall some definitions and some convergence results.

Definition 1. *We say that* G *and* $-\nabla F$ *the angle condition if there exists a real number* $\alpha > 0$ *such that*

$$\langle G(u), -\nabla F(u)\rangle \geq \alpha \|G(u)\| \|\nabla F(u)\|, \qquad u \in \Omega \subset \mathbb{R}^n, \tag{2}$$

where $\langle .,. \rangle$ *denotes the normal Eulidean inner product in* $\mathbb{R}^n$.

Definition 2. *We say that* G *and* $-\nabla F$ *the comparability condition if there exists a real number* $\alpha > 0$ *such that*

$$\alpha \|G(u)\| \geq \|\nabla F(u)\| \geq \alpha^{-1} \|G(u)\|, \qquad u \in \Omega. \tag{3}$$

Of course, if G and $-\nabla F$ satisfy the angle and comparability condition then F is a strict Lyapunov function.

Definition 3. *We denote by* $\Im$ *the class of non-decreasing function* $\Theta \in C(\mathbb{R}^+, \mathbb{R}^+)$ *such that*

$$\Theta(0) = 0, \quad \Theta > 0 \text{ on } (0, +\infty), \quad 1/\Theta \in L^1_{loc}(\mathbb{R}^+). \tag{4}$$

Definition 4. Let $a \in \Omega$.

1) We say that the function F *satisfies a Lojasiewicz inequality at* a *if there exists* $\theta \in (0, 1/2]$, $\gamma > 0$ *and* $\sigma > 0$ *such that*

$$|F(v) - F(a)|^{1-\theta} \leq \gamma \|\nabla F(v)\|, \quad \forall v \in B(a, \sigma) \cap \Omega. \tag{5}$$

The coefficient θ *is called a Lojasiewicz exponent.*

2) The function F *satisfies a Kurdyka-Lojasiewicz inequality at* a *if there exists* $\sigma > 0$ *and a function* $\Theta \in \Im$ *such that*

$$\Theta(|F(v) - F(a)|) \leq \|\nabla F(v)\|, \quad \forall v \in B(a, \sigma) \cap \Omega. \tag{6}$$

Notice that the first definition is a particular case of the second one with $\Theta(x) = (1/\gamma) x^{1-\theta}$.

In fact, the Lojasiewicz inequality comes from the following result which is proposed by Lojasiewicz in 1965.

Theorem 1. (Lojasiewicz [10], see also [13]).

If $F: \Omega \subset \mathbb{R}^n \to \mathbb{R}$ *is real analytic in some neighborhood of a point* $a \in \Omega$, *then* F *satisfies the Lojasiewicz inequality (5) at* a.

In the paper [11], Merlet and Nguyen proved the convergence result of the solution of gradient-like system (1) with Kurdyka-Lojasiewicz inequality. Then they obtained the convergence rate if F satisfies the Lojasiewicz inequality depending on the Lojasiewicz exponent. These results can be summarized by two following theorems, see Merlet & Nguyen [11].

Theorem 2.

Let u be a bounded global solution of (1) which admits strict Lyapunov function F. Assume that G, ∇F satisfy the angle condition (2) and F satisfies the Kurdyka-Lojasiewicz inequality (6). Then the gradient-like trajectory $u(t)$ converges to $a \in \omega[u]$ at infinity.

Theorem 3.

Under the hypotheses of Theorem 2, assume moreover that if G, ∇F satisfy the comparability condition (3) and F satisfies the Lojasiewicz inequality (5) then there exist some constants c, $\mu > 0$ such that

$$\left\|u(t) - \varphi\right\| \leq \begin{cases} ce^{-\mu t} & \text{if } \theta = 1/2, \\ ct^{-\theta/(1-2\theta)} & \text{if } 0 < \theta < 1/2 \end{cases}, \qquad \forall t \geq 0, \tag{7}$$

where θ is the Lojasiewicz exponent in (5).

Notice that if u is a bounded solution of (1) then the ω-limit set is non-empty, connect and compact subset of the critical points of F.

3. The main results

In this paper, we consider a growth condition of the form, namely a weak angle condition, as follows

$$\begin{cases} \left\langle G(u(t)), -\nabla F(u(t)) \right\rangle \geq \alpha \left\|G(u(t))\right\| \left\|\nabla F(u(t))\right\| + M'(t), & (8a) \\ \beta \left\|\nabla F(u)\right\| \geq \left\|G(u)\right\| + N'(t), & (8b) \end{cases}$$

for all $t \geq 0$ and some positive constants α, β and $M, N \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ are non increasing functions satisfying

$$\lim_{t \to \infty} M(t) = \lim_{t \to \infty} N(t) = 0. \tag{9}$$

This condition was proposed by Huang in [6,7].

However, the Kurdyka-Lojasiewicz inequality is not sufficient to prove the convergence result under this weak angle condition. In this case, we assume moreover that the function Θ satisfies the following hypothesis:

there exists a function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\Theta(x + y) \leq C\left[\Theta(x) + \Psi(y)\right], \quad \forall x, y \in \mathbb{R}^+, \tag{10}$$

for some constant $C > 0$.

Lemma 4.

The Lojasiewicz inequality (5) is also a particular case of the Kurdyka-Lojasiewicz inequality (6) and the function Θ satisfies the hypothesis (10) with $\Psi \equiv \Theta$ and $C = 1$.

Proof.

We need only to check that the function $\Theta(x) = x^{\theta}, \quad \theta \in [1/2, 1)$ satisfies (10). For $y \geq 0$ fixed, let us define

$$f_y(x) = (x+y)^{\theta} - x^{\theta} - y^{\theta}, \qquad x \geq 0.$$

We have

$$f_y'(x) = \theta \frac{x^{1-\theta} - (x+y)^{1-\theta}}{x^{1-\theta}(x+y)^{1-\theta}} \leq 0, \qquad \forall x > 0.$$

So the function f_y is non increasing. It follows that $f_y(x) \leq f_y(0) = 0$. The proof is complete. $\square$

Theorem 5 (Convergence result).

Let u be a bounded global solution of (1) which admits strict Lyapunov function F. Assume that G, ∇F satisfy the weak angle condition (8a,8b) and F satisfies the Kurdyka-Lojasiewicz inequality (6).

Assume moreover that there exists a function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that Θ, Ψ satisfy (10) and

$$\int_0^{\infty} \Psi(M(t))\, dt < \infty, \tag{11}$$

then $u' \in L^1(\mathbb{R}^+)$

$$\int_{\mathbb{R}^+} \|u'(t)\|\, dt < \infty, \tag{12}$$

and the gradient-like trajectory $u(t)$ converges to $a \in \omega[u]$ as t goes to infinity.

Proof.

We always keep in mind that the trajectory u is a global solution of (1). Then, using the growth condition (8a) and the fact that M is a non increasing function, we have

$$\begin{aligned} \frac{d}{dt}\left[F(u(t)) + M(t)\right] &= \langle G(u(t)), \nabla F(u(t)) \rangle + M'(t) \\ &\leq -\alpha \|G(u(t))\| \|\nabla F(u(t))\| \quad \leq \quad 0. \end{aligned} \tag{13}$$

This implies the function $F(u(t))+M(t)$ is also non increasing. Since we consider the bounded solution u, so there exists $a\in\omega[u]$. By continuity of F, M and $\lim_{t\to\infty} M(t)=0$, we obtain that $F(u(t))$ converges to $F(a)$ as t goes to infinity. Changing by an additive constant if necessary, we may assume $F(a)=0$, so that

$$F(u(t))+M(t)\geq 0.$$

If $F(u(t_0))+M(t_0)=0$ for some $t_0\geq 0$ then $F(u(t))+M(t)=0,$ for every $t\geq t_0$. By (13), there are two cases $G(u(t))=0$ or $\nabla F(u(t))=0$ for $t\geq t_0$. Of course we get (12) in the first case. In the second case, since F is a strict Lyapunov function, therefore $u(t)$ is constant for $t\geq t_0$. So there remains nothing to prove in these cases. Hence, without loss of generality we may assume $F(u(t))+M(t)>0,$ for every $t\geq 0.$

Since the function F satisfies the Kurdyka-Lojasiewicz inequality at a, there exists $\sigma>0$ and a function $\Theta\in\Im$ such that

$$\Theta\left(\left|F(v)-F(a)\right|\right)\leq\left\|\nabla F(v)\right\|,\quad \forall v\in B(a,\sigma)\cap\Omega.$$

In particular, since $a\in\omega[u]$ and $F(a)=0$, we obtain

$$\Theta\left(\left|F(u(t))\right|\right)\leq\left\|\nabla F(u(t))\right\|. \tag{14}$$

Let us define

$$\Phi(x)=\int_0^x \frac{1}{\Theta(s)}ds,\qquad x\geq 0,$$

and $E(t)=\Phi\left(F(u(t))+M(t)\right),\quad t\geq 0.$

Notice that the function Φ is well-defined by the above assumptions that $F(u(t))+M(t)>0$ and $1/\Theta\in L^1_{loc}(\mathbb{R}^+)$. On the other hand, it is easy to see that

$$\lim_{t\to\infty} E(t)=0.$$

Using the weak angle condition (8a) , we have

$$-E'(t)=\frac{\left\langle G(u(t)),-\nabla F(u(t))\right\rangle-M'(t)}{\Theta\left(F(u(t))+M(t)\right)}\geq\frac{\alpha\left\|G(u(t))\right\|\left\|\nabla F(u(t))\right\|}{\Theta\left(F(u(t))+M(t)\right)}.$$

Since Θ satisfies the condition (10), we get that

$$-E'(t)\geq\frac{\alpha}{C}.\frac{\left\|u'(t)\right\|\left\|\nabla F(u(t))\right\|}{\Theta\left(F(u(t))\right)+\Psi\left(M(t)\right)}. \tag{15}$$

The function F satisfies the Kurdyka-Lojasiewicz inequality at a, using (14), the estimate goes further as follows

$$-E'(t) \geq \frac{\alpha}{C} \cdot \frac{\|u'(t)\|\|\nabla F(u(t))\|}{\|\nabla F(u(t))\| + \Psi(M(t))} = \frac{\alpha}{C}\|u'(t)\| - \frac{\alpha}{C} \cdot \frac{\Psi(M(t))\|u'(t)\|}{\|\nabla F(u(t))\| + \Psi(M(t))}.$$

It follows that

$$\|u'(t)\| \leq -\frac{C}{\alpha} E'(t) + \frac{\Psi(M(t))\|u'(t)\|}{\|\nabla F(u(t))\| + \Psi(M(t))},$$

which, combined with the condition (8b), yields that

$$\|u'(t)\| \leq -\frac{C}{\alpha} E'(t) + \Psi(M(t)) \frac{\beta\|\nabla F(u(t))\| - N'(t)}{\|\nabla F(u(t))\| + \Psi(M(t))}. \tag{16}$$

On the other hand, since the function N is non increasing, so that $N'(t) \leq 0$, for every $t \geq 0$. Moreover, the function Ψ is positive. So we get two estimates

$$\frac{\|\nabla F(u(t))\|}{\|\nabla F(u(t))\| + \Psi(M(t))} \leq 1, \quad \text{and} \quad \frac{\Psi(M(t))}{\|\nabla F(u(t))\| + \Psi(M(t))} \leq 1.$$

Combining these estimates with (16), we have that

$$\|u'(t)\| \leq -\frac{C}{\alpha} E'(t) + \beta\Psi(M(t)) - N'(t). \tag{17}$$

Taking together with the assumption (11) and noting that $\lim_{t\to\infty} E(t) = \lim_{t\to\infty} N(t) = 0$, we obtain

$$\int_0^\infty \|u'(t)\| dt \leq \frac{C}{\alpha} E(0) + \beta \int_0^\infty \Psi(M(t)) dt + N(0) < \infty .$$

This is complete the proof. □

As in the paper of Merlet and Nguyen [11], in order to obtain the convergence rate, we need to consider the function F satisfying a Lojasiewicz inequality.

Lemma 6.

Let $\mu > 0$, $\alpha \geq 1$ and y be a solution of the following ordinary differential equation

$$y'(t) + \mu(y(t))^\alpha \leq 0, \quad t \geq 0,$$

Then for t large enough, we have

$$y(t) \leq \begin{cases} ce^{-\mu t} & if \quad \alpha = 1, \\ ct^{-1/(\alpha-1)} & if \quad \alpha > 1. \end{cases}$$

Proof.

In the case $\alpha = 1$, we get $y'(t) + \mu y(t) \le 0$.

Writing $y(t) = e^{-\mu t} g(t)$, we conclude that g is non increasing, so $y(t) \le e^{-\mu t} g(0) = ce^{-\mu t}$, for every $t \ge 0$.

In the second case $\alpha > 1$, we set $g(t) := (y(t))^{1-\alpha}$.

This function satisfies $g'(t) \ge \frac{\mu}{\alpha - 1}$, which implies $g(t) \ge ct$ for t large enough. Hence $y(t) \le ct^{-1/(\alpha-1)}$. □

Theorem 7 (Convergence rates).

Under the hypothesis of Theorem 5, assume moreover that F satisfies a Lojasiewicz inequality (5) with a Lojasiewicz exponent $\theta \in (0, 1/2]$ and

$$\|G(u(t))\| \ge \lambda \|\nabla F(u(t))\| + K'(t), \tag{18}$$

Where $K \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ is non increasing function and $\lim_{t \to \infty} K(t) = 0$.

Then there exists a constant $c > 0$ such that

$$\|u(t) - a\| \le c\left(v(t) + w(t) + \int_t^{\infty} \Psi(M(t))\, dt + N(t) \right), \tag{19}$$

where v is a solution of the ordinary differential equation

$$v'(t) + c\alpha.(v(t))^{\frac{1-\theta}{\theta}} = \alpha(\lambda + \beta)\Theta(M(t)) - \alpha N'(t) - \alpha K'(t),$$

and $w(t) := \begin{cases} ce^{-c\alpha t} & \text{if } \theta = 1/2, \\ ct^{-\theta/(1-2\theta)} & \text{if } 0 < \theta < 1/2. \end{cases}$

Remark.

With the hypotheses in Theorem 7, the right hand side of (19) converges to 0 as t goes to infinity.

Proof.

In the previous proof, we know that $u(t)$ converges to a as t goes to infinity. Let us use the same notation with the proof of Theorem 5, by the estimate (17), we obtain that

$$\begin{aligned}\|u(t)-a\| &\le \int_t^\infty \|u'(t)\|dt \le -\frac{C}{\alpha}\int_t^\infty (E'(t))dt + \beta\int_t^\infty \Psi(M(t))dt - \int_t^\infty (N'(t))dt \\ &= \frac{C}{\alpha}E(t) + \beta\int_t^\infty \Psi(M(t))dt + N(t).\end{aligned} \tag{20}$$

By using estimate (17) again and assumption (18), we get that

$$\lambda\|\nabla F(u(t))\| \le -\frac{C}{\alpha}E'(t) + \beta\Psi(M(t)) - N'(t) - K'(t).$$

Combining with the Lojasiewicz inequality, it follows that

$$\lambda\Theta(F(u(t))) \le -\frac{C}{\alpha}E'(t) + \beta\Psi(M(t)) - N'(t) - K'(t),$$

where the function Θ defined by $\Theta(x) = (1/\gamma)x^{1-\theta}, \quad x \ge 0.$ In this case, by Lemma 4, the function Ψ coincides to Θ and $C = 1$. This implies

$$\lambda\Theta(F(u(t))) \le -\frac{1}{\alpha}E'(t) + \beta\Theta(M(t)) - N'(t) - K'(t), \tag{21}$$

With the same definition as in the proof of Theorem 5, we have the explicit formula $\Phi(x) = \frac{\gamma}{\theta}x^{\theta}, \quad x \ge 0.$ As a consequence, we have

$$E(t) = \frac{\gamma}{\theta}(F(u(t)) + M(t))^{\theta},$$

and

$$\Theta(F(u(t)) + M(t)) = \frac{1}{\gamma}\left(\frac{\theta}{\gamma}\right)^{\frac{1-\theta}{\theta}}(E(t))^{\frac{1-\theta}{\theta}}.$$

Now let us estimate

$$\begin{aligned}\lambda\Theta(F(u(t))) &\ge \lambda\Theta(F(u(t)) + M(t)) - \lambda\Theta(M(t)) \\ &\ge c.(E(t))^{\frac{1-\theta}{\theta}} - \lambda\Theta(M(t)),\end{aligned} \tag{22}$$

where $c = \frac{\lambda}{\gamma}\left(\frac{\theta}{\gamma}\right)^{\frac{1-\theta}{\theta}} > 0.$ Taking the estimates (21) and (22) together, it implies

$$E'(t) + c\alpha.(E(t))^{\frac{1-\theta}{\theta}} \le \alpha(\lambda + \beta)\Theta(M(t)) - \alpha N'(t) - \alpha K'(t).$$

Let us set $h(t) := E(t) - v(t)$, where v is the solution of the following ordinary differential equation

$$v'(t) + c\alpha.(v(t))^{\frac{1-\theta}{\theta}} = \alpha(\lambda+\beta)\Theta(M(t)) - \alpha N'(t) - \alpha K'(t).$$

So we have

$$h'(t) + c\alpha.(h(t))^{\frac{1-\theta}{\theta}} \le 0.$$

By Lemma 6, we obtain the convergence rate as follows

$$E(t) - v(t) \le w(t) := \begin{cases} ce^{-c\alpha t} & if \quad \theta = 1/2, \\ ct^{-\theta/(1-2\theta)} & if \quad 0 < \theta < 1/2. \end{cases} \tag{23}$$

Combining (20) and (23), we finish the proof. □

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