STRONG CONVERGENCE OF INERTIAL HYBRID ITERATION FOR TWO ASYMPTOTICALLY $G$-NONEXPANSIVE MAPPINGS IN HILBERT SPACE WITH GRAPHS

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ABSTRACT

In this paper, by combining the shrinking projection method with a modified inertial S-iteration process, we introduce a new inertial hybrid iteration for two asymptotically $G$-nonexpansive mappings and a new inertial hybrid iteration for two $G$-nonexpansive mappings in Hilbert spaces with graphs. We establish a sufficient condition for the closedness and convexity of the set of fixed points of asymptotically $G$-nonexpansive mappings in Hilbert spaces with graphs. We then prove a strong convergence theorem for finding a common fixed point of two asymptotically $G$-nonexpansive mappings in Hilbert spaces with graphs. By this theorem, we obtain a strong convergence result for two $G$-nonexpansive mappings in Hilbert spaces with graphs. These results are generalizations and extensions of some convergence results in the literature, where the convexity of the set of edges of a graph is replaced by coordinate-convexity. In addition, we provide a numerical example to illustrate the convergence of the proposed iteration processes.

Keywords: asymptotically $G$-nonexpansive mapping; Hilbert space with graphs; inertial hybrid iteration

1. Introduction and preliminaries

In 2012, by using the combination concepts between the fixed point theory and the graph theory, Aleomraninejad, Rezapour, and Shahzad (2012) introduced the notions of $G$-contractive mapping and $G$-nonexpansive mapping in a metric space with directed graphs and stated the convergence for these mappings. After that, there were many convergence results for $G$-nonexpansive mappings by some iteration processes established in Hilbert spaces and Banach spaces with graphs. In 2018, Sangago, Hunde, and Hailu (2018) introduced the notion of an asymptotically $G$-nonexpansive mapping and proved the weak and strong convergence of a modified Noor iteration process to common fixed points of a finite family of asymptotically $G$-nonexpansive mappings in Banach spaces with graphs. After that some authors proposed a two-step iteration process for two asymptotically $G$-nonexpansive mappings $T_1, T_2 : \Omega \rightarrow \Omega$ (Wattanataweekul, 2018) and a three-step iteration process for three asymptotically $G$-nonexpansive mappings $T_1, T_2, T_3 : \Omega \rightarrow \Omega$ (Wattanataweekul, 2019) as follows:

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where \( \{a_n\}, \{b_n\}, \{c_n\} \subset [0, 1] \). Furthermore, the authors also established the weak and strong convergence results of the iteration process (1.1) and the iteration process (1.2) to common fixed points of asymptotically \( G \)-nonexpansive mappings in Banach spaces with graphs.

Currently, there were many methods to construct new iteration processes which generalize some previous iteration processes. In 2008, Mainge proposed the inertial Mann iteration by combining the Mann iteration and the inertial term \( \gamma_n (u_n - u_{n-1}) \). In 2018, by combining the CQ-algorithm and the inertial term, Dong, Yuan, Cho, and Rassias (2018) studied an inertial CQ-algorithm for a non-expansive mapping as follows:

\[
\begin{align*}
\{w_n = u_n + \gamma_n (u_n - u_{n-1}) \\
v_n = (1 - a_n) w_n + a_n T w_n \\
u_{n+1} = P_{c_n Q_n} u_n,
\end{align*}
\]

where \( \{a_n\} \subset [0, 1], \{\gamma_n\} \subset [\alpha, \beta] \) for some \( \alpha, \beta \in \mathbb{R} \), \( T : H \to H \) is a nonexpansive mapping, and \( P_{c_n Q_n} u_n \) is the metric projection of \( u_n \) onto \( C_n \cap Q_n \).

In 2019, by combining a modified S-iteration process with the inertial extrapolation, Phon-on, Makaje, Sama-Ae, and Khongraphan (2019) introduced an inertial S-iteration process for two nonexpansive mappings such as:

\[
\begin{align*}
\{w_n = u_n + \gamma_n (u_n - u_{n-1}) \\
v_n = (1 - a_n) w_n + a_n T w_n \\
u_{n+1} = (1 - b_n) T w_n + b_n T v_n,
\end{align*}
\]

where \( \{a_n\}, \{b_n\} \subset [0, 1], \{\gamma_n\} \subset [\alpha, \beta] \) for some \( \alpha, \beta \in \mathbb{R} \), and \( T_1, T_2 : H \to H \) are two nonexpansive mappings. Recently, by combining the shrinking projection method with a modified S-iteration process, Hammad, Cholamjiak, Yambangwai, and Dutta (2019) introduced the following hybrid iteration for two \( G \)-nonexpansive mappings

\[
\begin{align*}
\{v_n = (1 - b_n) u_n + b_n T v_n \\
w_n = (1 - a_n) T v_n + a_n T^2 v_n \\
\Omega_{n+1} = \{w \in \Omega_n : || w_n - w || \leq || u_n - w || \}
\end{align*}
\]

where \( \{a_n\}, \{b_n\} \subset [0, 1], T_1, T_2 : \Omega \to \Omega \) are two \( G \)-nonexpansive mappings, and \( P_{\Omega_{n+1}} u_n \) is the metric projection of \( u_n \) onto \( \Omega_{n+1} \).
Motivated by these works, we introduce an iteration process for two $G$-nonexpansive mappings $T_1, T_2 : H \to H$ such as:

$$
\begin{align*}
&z_n = u_n + \gamma_n (u_n - u_{n-1}) \\
v_n = (1 - b_n) z_n + b_n T_n z_n \\
w_n = (1 - a_n) T_1 v_n + a_n T_2 z_n \\
\Omega_{n+1} = \{ w \in \Omega_n : \| w - w \| \leq \| z_n - w \| \}
\end{align*}
$$

and an iteration process for two asymptotically $G$-nonexpansive mappings $T_1, T_2 : H \to H$ such as:

$$
\begin{align*}
&z_n = u_n + \gamma_n (u_n - u_{n-1}) \\
v_n = (1 - b_n) z_n + b_n T_1^n z_n \\
w_n = (1 - a_n) T_1^n v_n + a_n T_2^n z_n \\
\Omega_{n+1} = \{ w \in \Omega_n : \| w - w \|^p \leq \| z_n - w \|^p + \epsilon_n \}
\end{align*}
$$

where $\{a_n\}, \{b_n\} \subset [0, 1], \{\gamma_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in \mathbb{R}, H$ is a real Hilbert space, $P_{\Omega_{n+1}} u_n$ is the metric projection of $u_n$ onto $\Omega_{n+1}$, and $\epsilon_n$ is defined in Theorem 2.2 in Section 2. Then, under some conditions, we prove that the sequence $\{u_n\}$ generated by (1.5) strongly converges to the projection of the initial point $u_1$ onto the set of all common fixed points of $T_1$ and $T_2$ in Hilbert spaces with graphs. By this theorem, we obtain a strong convergence result for two $G$-nonexpansive mappings by the iteration process (1.4) in Hilbert spaces with graphs. In addition, we give a numerical example for supporting obtained results.

We now recall some notions and lemmas as follows:

Throughout this paper, let $G = (V(G), E(G))$ be a directed graph, where the set all vertices and edges denoted by $V(G)$ and $E(G)$, respectively. We assume that all directed graphs are reflexive, that is, $(u, u) \in E(G)$ for each $u \in V(G)$, and $G$ has no parallel edges. A directed graph $G = (V(G), E(G))$ is said to be transitive if for any $u, v, w \in V(G)$ such that $(u, v)$ and $(v, w)$ are in $E(G)$, then $(u, w) \in E(G)$.

**Definition 1.1.**

Tiammee, Kaewkhao, & Suantai (2015, p.4): Let $X$ be a normed space, $\Omega$ be a nonempty subset of $X$, and $G = (V(G), E(G))$ be a directed graph such that $V(G) = \Omega$. Then $\Omega$ is said to have property $(G)$ if for any sequence $\{u_n\}$ in $\Omega$ such that $(u_n, u_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $\{u_n\}$ weakly converging to $u \in \Omega$, then there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $(u_{n(k)}, u) \in E(G)$ for all $k \in \mathbb{N}$.

**Definition 1.2.**

Nguyen, & Nguyen (2020): Definition 3.1: Let $X$ be a normed space and $G = (V(G), E(G))$ be a directed graph such that $E(G) \subset X \times X$. The set of edges $E(G)$ is said to be coordinate-convex if for all $(p, u), (p, v), (u, p), (v, p) \in E(G)$ and for all $t \in [0, 1]$, then $t(p, u) + (1 - t)(p, v) \in E(G)$ and $t(u, p) + (1 - t)(v, p) \in E(G)$. 

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Definition 1.3.
Tripak (2016) - Definition 2.1 and Sangago et al. (2018)- Definition 3.1: Let $X$ be a normed space, $G = (V(G), E(G))$ be a directed graph such that $V(G) \subset X$ and $T : V(G) \to V(G)$ be a mapping. Then

(1) $T$ is said to be $G$-nonexpansive if
(a) $T$ is edge-preserving, that is, for all $(u, v) \in E(G)$, we have $(Tu, Tv) \in E(G)$.
(b) $|| Tu - Tv || \leq || u - v ||$, whenever $(u, v) \in E(G)$ for any $u, v \in V(G)$.

(2) $T$ is call asymptotically $G$-nonexpansive mapping if
(a) $T$ is edge-preserving.
(b) There exists a sequence $\{\lambda_n\} \subset [1, \infty)$ with $\sum_{n=1}^{\infty} (\lambda_n - 1) < \infty$ such that $|| T^n u - T^n v || \leq \lambda_n || u - v ||$ for all $n \in \mathbb{N}$, whenever $(u, v) \in E(G)$ for any $u, v \in V(G)$, where $\{\lambda_n\}$ is said to be an asymptotic coefficient sequence.

Remark 1.4.
Every $G$-nonexpansive mapping is an asymptotically $G$-nonexpansive mapping with the asymptotic coefficients $\lambda_n = 1$ for all $n \in \mathbb{N}$.

Lemma 1.5.
Sangago et al. (2018) - Theorem 3.3: Let $\Omega$ be a nonempty closed, convex subset of a real Banach space $X$, $G = (V(G), E(G))$ be a directed graph such that $V(G) = \Omega$, $T : \Omega \to \Omega$ be an asymptotically $G$-nonexpansive mapping, $\{u_n\}$ be a sequence in $\Omega$ converging weakly to $u \in \Omega$, $(u_n, u_{n+1}) \in E(G)$ and $\lim_{n \to \infty} || Tu_n - u_n || = 0$. Then $Tu = u$.

Let $H$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $|| \cdot ||$. $\Omega$ be a nonempty, closed and convex subset of a Hilbert space $H$. Now, we recall some basic notions of Hilbert spaces which we will use in the next section.

The nearest point projection of $H$ onto $\Omega$ is denoted by $P_{\Omega}$, that is, for all $u \in H$, we have $|| u - P_{\Omega} u || = \inf\{ || u - v || : v \in \Omega \}$. Then $P_{\Omega}$ is called the metric projection of $H$ onto $\Omega$. It is known that for each $u \in H$, $p = P_{\Omega} u$ is equivalent to $\langle u - p, p - v \rangle \geq 0$ for all $v \in \Omega$.

Lemma 1.6.
Alber (1996, p.5): Let $H$ be a real Hilbert space, $\Omega$ be a nonempty, closed and convex subset of $H$, and $P_{\Omega}$ is the metric projection of $H$ onto $\Omega$. Then for all $u \in H$ and $v \in \Omega$, we have $|| v - P_{\Omega} u ||^2 + || u - P_{\Omega} u ||^2 \leq || u - v ||^2$.

Lemma 1.7.
Bauschke and Combettes (2011)- Corollary 2.14: Let $H$ be a real Hilbert space. Then for all $\lambda \in [0, 1]$ and $u, v \in H$, we have $|| \lambda u + (1 - \lambda)v ||^2 = \lambda || u ||^2 + (1 - \lambda) || v ||^2 - \lambda(1 - \lambda) || u - v ||^2$.

Lemma 1.8.
Martinez-Yanes and Xu (2006) – Lemma 13: Let $H$ be a real Hilbert space and $\Omega$ be a nonempty, closed and convex subset of $H$. Then for $x, y, z \in H$ and $a \in \mathbb{R}$, the following set is convex and closed: $\{ w \in \Omega : || y - w ||^2 \leq || x - w ||^2 + \langle z, w \rangle + a \}$. 

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The following result will be used in the next section. The proof of this lemma is easy and is omitted.

**Lemma 1.9.**

Let $H$ be a real Hilbert space. Then for all $u, v, w \in H$, we have

$$|| u - v ||^2 = || u - w ||^2 + || w - v ||^2 + 2 \langle u - w, w - v \rangle.$$ 

2. **Main results**

First, we denote by $F(T) = \{ u \in H : Tu = u \}$ the set of fixed points of the mapping $T : H \to H$. The following result is a sufficient condition for the closedness and convexity of the set $F(T)$ in real Hilbert spaces, where $T$ is an asymptotically $G$-nonexpansive mapping.

**Proposition 2.1.**

Let $H$ be a real Hilbert space, $G = (V(G), E(G))$ be a directed graph such that $V(G) = H$, $T : H \to H$ be an asymptotically $G$-nonexpansive mapping with an asymptotic coefficient sequence $\{ \lambda_n \} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty} (\lambda_n - 1) < \infty$, and $F(T) \times F(T) \subset E(G)$. Then

1. If $H$ have property $(G)$, then $F(T)$ is closed.
2. If the graph $G$ is transitive, $E(G)$ is coordinate-convex, then $F(T)$ is convex.

**Proof.**

(1). Suppose that $F(T) \neq \emptyset$. Let $\{ p_n \}$ be a sequence in $F(T)$ such that $\lim_{u \to \infty} || p_n - p || = 0$ for some $p \in H$. Since $F(T) \times F(T) \subset E(G)$, we have $(p_n, p_{n+1}) \in E(G)$. By combining this with property $(G)$ of $H$, we conclude that there exists a subsequence $\{ p_{n(k)} \}$ of $\{ p_n \}$ such that $(p_{n(k)}, p) \in E(G)$ for $k \in \mathbb{N}$. Since $T$ is an asymptotically $G$-nonexpansive mapping, we obtain

$$|| p - Tp || \leq || p - p_{n(k)} || + || Tp_{n(k)} - Tp || \leq (1 + \lambda_k) || p - p_{n(k)} ||.$$ 

It follows from the above inequality and $\lim_{u \to \infty} || p_n - p || = 0$ that $Tp = p$, that is, $p \in F(T)$. Therefore, $F(T)$ is closed.

(2). Let $p_1, p_2 \in F(T)$. For $t \in [0,1]$, we put $p = tp_1 + (1-t)p_2$. Since $F(T) \times F(T) \subset E(G)$ and $p_1, p_2 \in F(T)$, we get $(p_1, p) = (p, p_1), (p_2, p) = (p, p_2), (p_2, p_1) \in E(G)$. By combining this with $E(G)$ is coordinate-convex, we conclude that $t(p_1, p_1) + (1-t)(p_1, p_2) = (p, p) \in E(G)$, $t(p, p_1) + (1-t)(p, p_2) = (p, p_1) \in E(G)$ and $t(p_2, p_1) + (1-t)(p_2, p_2) = (p, p) \in E(G)$. Due to the fact that $T$ is an asymptotically $G$-nonexpansive mapping, for each $i = 1, 2$, we get

$$|| p_i - T^n p || \leq || T^n p_i - T^n p || \leq \lambda_n || p_i - p ||.$$ 

(2.1)

Furthermore, by using Lemma 1.9, we get

$$|| p_1 - T^n p ||^2 = || p_1 - p ||^2 + || p - T^n p ||^2 + 2 \langle p_1 - p, p - T^n p \rangle$$

(2.2)

and

$$|| p_2 - T^n p ||^2 = || p_2 - p ||^2 + || p - T^n p ||^2 + 2 \langle p_2 - p, p - T^n p \rangle.$$ 

(2.3)

It follows from (2.1) and (2.2) that
Also, we conclude from (2.1) and (2.3) that
\[ || p - T^n p ||^2 \leq \left( \lambda_n^2 - 1 \right) || p_1 - p ||^2 - 2(p_1 - p, p - T^n p). \]  
(2.5)
By multiplying $t$ on the both sides of (2.4), and multiplying $(1 - t)$ on the both sides of (2.5), we get
\[ || p - T^n p ||^2 \leq t(\lambda_n^2 - 1) || p_1 - p ||^2 + (1 - t) \left( \lambda_n^2 - 1 \right) || p_2 - p ||^2 \]
\[ = t(\lambda_n^2 - 1) || p_1 - p ||^2 + (1 - t) \left( \lambda_n^2 - 1 \right) || p_2 - p ||^2. \]  
(2.6)
Since $\sum_{n=1}^{\infty} (\lambda_n - 1) < \infty$, we have $\lim_{n \to \infty} \lambda_n = 1$. Therefore, from (2.6), we find that
\[ \lim_{n \to \infty} || p - T^n p || = 0. \]  
(2.7)
Furthermore, since $(p_1, p) \in E(G)$ and $T^n$ is edge-preserving, we have $(p_1, T^n p) \in E(G)$. Then, by the transitive property of $G$ and $(p, p_1), (p_1, T^n p) \in E(G)$, we get $(p, T^n p) \in E(G)$. Due to asymptotically $G$-nonexpansiveness of $T$, we obtain
\[ || Tp - p || \leq || Tp - T^{n+1} p || + || T^{n+1} p - p || \leq \lambda_n || p - T^n p || + || T^{n+1} p - p ||. \]  
(2.8)
Taking the limit in (2.8) as $n \to \infty$ and using (2.7), we find that $Tp = p$, that is, $p \in F(T)$. Therefore, $F(T)$ is convex.

Let $T_1, T_2 : H \to H$ be two asymptotically $G$-nonexpansive mappings with asymptotic coefficient sequences $\{\alpha_n\}, \{\beta_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty} (\alpha_n - 1) < \infty$ and $\sum_{n=1}^{\infty} (\beta_n - 1) < \infty$. Put $\lambda_n = \max\{\alpha_n, \beta_n\}$, we have $\{\lambda_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty} (\lambda_n - 1) < \infty$ and for all $(u, v) \in E(G)$ and for each $i = 1, 2$, we have $|| T_i^n u - T_i^n v || \leq \lambda_n || u - v ||$. In the following theorem, we also assume that $F = F(T_1) \cap F(T_2)$ is nonempty and bounded in $H$, that is, there exists a positive number $\kappa$ such that $F \subset \{u \in H : || u || \leq \kappa\}$. The following result shows the strong convergence of iteration process (1.5) to common fixed points of two asymptotically $G$-nonexpansive mappings in Hilbert spaces with directed graphs. 

**Theorem 2.2.**

Let $H$ be a real Hilbert space, $H$ have property $(G)$, $G = (V(G), E(G))$ be a directed transitive graph such that $V(G) = H$, $E(G)$ be coordinate-convex, $T_1, T_2 : H \to H$ be two asymptotically $G$-nonexpansive mappings such that $F(T_1) \times F(T_2) \subset E(G)$ for all $i = 1, 2$, $\{u_n\}$ be a sequence generated by (1.5) where $\{a_n\}, \{b_n\}$ are sequences in $[0,1]$ such that $0 < \lim_{n \to \infty} a_n \leq \limsup_{n \to \infty} a_n < 1$, $0 < \liminf_{n \to \infty} b_n \leq \limsup_{n \to \infty} b_n < 1$, and $\gamma_n \in [\alpha, \beta]$ for some $\alpha, \beta \in \mathbb{R}$.
such that \((u_n, p), (p, u_n), (z_n, p) \in E(G)\) for all \(p \in F\); \(\varepsilon_n = (\lambda_1^2 - 1)(1 + b_n \lambda_n^2)(|| z_n || + \kappa)^2\). Then the sequence \(\{u_n\}\) strongly converges to \(P_F u_i\).

**Proof.**

The proof of Theorem 2.2 is divided into six steps.

**Step 1.** We show that \(P_F u_i\) is well-defined. Indeed, by Proposition 2.1, we conclude that \(F(T_1)\) and \(F(T_2)\) are closed and convex. Therefore, \(F = F(T_1) \cap F(T_2)\) is closed and convex. Note that \(F\) is nonempty by the assumption. This fact ensures that \(P_F u_i\) is well-defined.

**Step 2.** We show that \(P_{\Omega_{n+1}} u_i\) is well-defined. We first prove by a mathematical induction that \(\Omega_n\) is closed and convex for \(n \in \mathbb{N}\). Obviously, \(\Omega_1 = H\) is closed and convex. Now we suppose that \(\Omega_n\) is closed and convex. Then by the definition of \(\Omega_{n+1}\) and Lemma 1.8, we conclude that \(\Omega_{n+1}\) is closed and convex. Therefore, \(\Omega_n\) is closed and convex for \(n \in \mathbb{N}\).

Next, we show that \(F \subset \Omega_{n+1}\) for all \(n \in \mathbb{N}\). Indeed, for \(p \in F\), we have \(T_1 p = T_2 p = p\). Since \((z_n, p) \in E(G)\) and \(T_i^*\) is edge-preserving, we obtain \((T_1^* z_n, p) \in E(G)\). Due to the coordinate-convexity of \(E(G)\), we get

\[
(\nu_1, p) = (1 - b_n)(z_n, p) + b_n(T_1^* z_n, p) \quad \in \quad E(G).
\]

It follows from Lemma 1.7 and asymptotically \(G\)-nonexpansiveness of \(T_1, T_2\) that

\[
|| w_n - p ||^2 = || (1 - a_n)(T_1^* v_n - p) + a_n(T_2^* v_n - p) ||^2
\]

\[
= (1 - a_n) || T_1^* v_n - p ||^2 + a_n || T_2^* v_n - p ||^2 - a_n(1 - a_n) || T_2^* v_n - T_1^* v_n ||^2
\]

\[
\leq (1 - a_n) \lambda_n^2 || v_n - p ||^2 + a_n \lambda_n^2 || v_n - p ||^2 - a_n(1 - a_n) || T_2^* v_n - T_1^* v_n ||^2
\]

\[
= \lambda_n^2 || v_n - p ||^2 - a_n(1 - a_n) || T_2^* v_n - T_1^* v_n ||^2
\]

\[
\leq \lambda_n^2 || v_n - p ||^2 - a_n(1 - a_n) || T_2^* v_n - T_1^* v_n ||^2
\]

\[
\leq \lambda_n^2 || v_n - p ||^2 - a_n(1 - a_n) || T_2^* v_n - T_1^* v_n ||^2
\]

\[
(2.9)
\]

and

\[
|| v_n - p ||^2 = || (1 - b_n)(z_n - p) + b_n(T_1^* z_n - p) ||^2
\]

\[
= (1 - b_n) || z_n - p ||^2 + b_n || T_1^* z_n - p ||^2 - b_n(1 - b_n) || T_1^* z_n - z_n ||^2
\]

\[
\leq (1 - b_n) || z_n - p ||^2 + b_n \lambda_n^2 || z_n - p ||^2 - b_n(1 - b_n) || T_1^* z_n - z_n ||^2
\]

\[
\leq 1 + b_n(\lambda_n^2 - 1) || z_n - p ||^2 - b_n(1 - b_n) || T_1^* z_n - z_n ||^2
\]

\[
\leq 1 + b_n(\lambda_n^2 - 1) || z_n - p ||^2 .
\]

\[
(2.10)
\]

By substituting (2.10) into (2.9), we obtain

\[
|| w_n - p ||^2 \leq \lambda_n^2[1 + b_n(\lambda_n^2 - 1)] || z_n - p ||^2
\]

\[
\leq || z_n - p ||^2 + (\lambda_n^2 - 1)(1 + b_n \lambda_n^2)(|| z_n || || p || + \kappa)^2
\]

\[
\leq || z_n - p ||^2 + (\lambda_n^2 - 1)(1 + b_n \lambda_n^2)(|| z_n || + \kappa)^2
\]

\[
= || z_n - p ||^2 + \varepsilon_n .
\]

\[
(2.11)
\]

It follows from (2.11) that \(p \in \Omega_{n+1}\) and hence \(F \subset \Omega_{n+1}\) for all \(n \in \mathbb{N}\). Since \(F \neq \emptyset\), we have \(\Omega_{n+1} = \emptyset\) for all \(n \in \mathbb{N}\). Therefore, we find that \(P_{\Omega_{n+1}} u_i\) is well-defined.
Step 3. We show that \( \lim_{n \to \infty} || u_n - u_1 || \) exists. Indeed, since \( u_n = P_{\Omega_n} u_1 \), we have
\[ || u_n - u_1 || \leq || x - u_1 || \quad \text{for all} \quad x \in \Omega_n. \] (2.12)
Since \( u_{n+1} = P_{\Omega_{n+1}} u_1 \in \Omega_{n+1} \subseteq \Omega_n \), by taking \( x = u_{n+1} \) in (2.12), we obtain
\[ || u_n - u_1 || \leq || u_{n+1} - u_1 ||. \]

Since \( F \) is nonempty, closed and convex subset of \( H \), there exists a unique \( q = P_F u_1 \) and hence \( q \in F \subseteq \Omega_n \). Therefore, by choosing \( x = q \) in (2.12), we get
\[ || u_n - u_1 || \leq || q - u_1 ||. \]
By the above, we conclude that the sequence \( \{ || u_n - u_1 || \} \) is bounded and nondecreasing. Therefore, \( \lim_{n \to \infty} || u_n - u_1 || \) exists.

Step 4. We show that \( \lim_{n \to \infty} u_n = u \) for some \( u \in H \). Indeed, it follows from \( u_n = P_{\Omega_n} u_1 \) and Lemma 1.6, we get
\[ || v - u_n ||^2 + || u_1 - u_n ||^2 \leq || v - u_1 ||^2 \quad \text{for all} \quad v \in \Omega_n. \] (2.13)
For \( m > n \), we see that \( u_m = P_{\Omega_m} u_1 \in \Omega_m \subseteq \Omega_n \). By taking \( v = u_m \) in (2.13), we have
\[ || u_n - u_m ||^2 + || u_1 - u_m ||^2 \leq || u_n - u_1 ||^2. \]
This implies that \( || u_n - u_m ||^2 \leq \| u_n - u_1 \|^2 - \| u_m - u_1 \|^2 \).

It follows from the above inequality and the existence of \( \lim_{n \to \infty} || u_n - u_1 || \) that
\[ \lim_{m,n \to \infty} || u_m - u_n || = 0 \] and hence \( \{ u_n \} \) is a Cauchy sequence. Therefore, there exists \( u \in H \) such that \( \lim_{n \to \infty} u_n = u \). Moreover, we also have
\[ \lim_{n \to \infty} || u_{n+1} - u_n || = 0. \] (2.14)

Step 5. We show that \( u \in F \). Indeed, since \( u_{n+1} \in \Omega_n \), by the definition of \( \Omega_{n+1} \), we get
\[ || w_n - u_{n+1} ||^2 \leq || z_n - u_{n+1} ||^2 + \varepsilon_n \] (2.15)

It follows from \( || z_n - u_n || = \gamma_n \cdot || u_n - u_{n-1} || \) and (2.14) that
\[ \lim_{n \to \infty} || z_n - u_n || = 0. \] (2.16)
Therefore, we conclude from (2.14) and (2.16) that
\[ \lim_{n \to \infty} || z_n - u_{n+1} || = 0. \] (2.17)

It follows from (2.17) and the boundedness of the sequence \( \{ u_n \} \) that \( \{ z_n \} \) is bounded. Thus, there exists \( A > 0 \) such that
\[ 0 \leq \varepsilon_n = (\lambda_n^2 - 1)(1 + b_n \lambda_n^2)(|| z_n || + \kappa)^2 \leq A(\lambda_n^2 - 1). \]
Taking the limit in the above inequality as \( n \to \infty \) and using \( \lim_{n \to \infty} \lambda_n = 1 \), we get \( \lim_{n \to \infty} \varepsilon_n = 0 \). Then, by combining this with (2.15) and (2.17), we have
\[ \lim_{n \to \infty} || w_n - u_{n+1} || = 0. \] (2.18)

It follows from (2.14) and (2.18) that
\[ \lim_{n \to \infty} || w_n - u_n || = 0. \] (2.19)

Then by combining (2.16) and (2.19), we obtain that
\[ \lim_{n \to \infty} \| z_n - w_n \| = 0. \] \tag{2.20}

Next, for \( p \in F \), by the same proof of (2.9), (2.10) and (2.11), we get
\[
\| w_n - p \|^2 \leq \lambda_n [1 + b_n (\lambda_n^2 - 1)] \| z_n - p \|^2 + \lambda_n b_n (1 - b_n) \| T_1^n z_n - z_n \|^2
\leq \| z_n - p \|^2 + \epsilon_n + b_n (1 - b_n) \| T_1^n z_n - z_n \|^2. \tag{2.21}
\]

It follows from (2.19) and the boundedness of the sequence \( \{u_n\} \) that \( \{w_n\} \) is bounded. Moreover, by the boundedness of \( \{z_n\} \) and \( \{w_n\} \), we conclude that there exists \( A_2 > 0 \) such that \( \| z_n \| + \| w_n \| \leq A_2 \) for all \( n \in \mathbb{N} \). It follows from (2.21) that
\[
b_n (1 - b_n) \| T_1^n z_n - z_n \|^2 \leq \| z_n - p \|^2 - \| w_n - p \|^2 + \epsilon_n
= \| z_n \|^2 - \| w_n \|^2 + 2(w_n - z_n, p) + \epsilon_n
\leq (\| z_n \| - \| w_n \|)(\| z_n \| + \| w_n \|) + 2 \| w_n - z_n \| \cdot \| p \| + \epsilon_n
\leq A_2 \| z_n - w_n \| + 2 \| w_n - z_n \| \cdot \| p \| + \epsilon_n. \tag{2.22}
\]

Therefore, by combining (2.22) with (2.20) and using \( \lim_{n \to \infty} \epsilon_n = 0 \), \( \lim_{n \to \infty} b_n (1 - b_n) > 0 \), we get
\[
\lim_{n \to \infty} \| T_1^n z_n - z_n \| = 0. \tag{2.23}
\]

Then by \((z_n, p), (p, u_n) \in E(G)\) and the transitive property of \( G \), we obtain \((z_n, u_n) \in E(G)\). Since \( T_1 \) is asymptotically \( G \)-nonexpansive and \((z_n, u_n) \in E(G)\), we get
\[
\| T_1^n u_n - u_n \| \leq \| T_1^n u_n - T_1^n z_n \| + \| T_1^n z_n - z_n \| + \| z_n - u_n \|
\leq \lambda_n \| u_n - z_n \| + \| T_1^n z_n - z_n \| + \| z_n - u_n \|
= (1 + \lambda_n) \| z_n - u_n \| + \| T_1^n z_n - z_n \|. \tag{2.24}
\]

It follows from (2.16), (2.23) and (2.24) that
\[
\lim_{n \to \infty} \| T_1^n u_n - u_n \| = 0. \tag{2.25}
\]

Next, by using similar argument as in the proof of (2.9), (2.10) and (2.11), we also obtain
\[
\| w_n - p \|^2 \leq \lambda_n [1 + b_n (\lambda_n^2 - 1)] \| z_n - p \|^2 + \lambda_n b_n (1 - b_n) \| T_1^n v_n - T_1^n v_n \|^2. \tag{2.26}
\]

By the same proof of (2.22), from (2.26) and \( \lim_{n \to \infty} \lambda_n (1 - \lambda_n) > 0 \), we get
\[
\lim_{n \to \infty} \| T_1^n v_n - T_1^n v_n \| = 0. \tag{2.27}
\]

It follows from \( \| v_n - z_n \| = b_n \| T_1^n z_n - z_n \| \) and (2.23) that
\[
\lim_{n \to \infty} \| v_n - z_n \| = 0. \tag{2.28}
\]

Then by combining (2.16) and (2.28), we have
\[
\lim_{n \to \infty} \| u_n - v_n \| = 0. \tag{2.29}
\]

Now, by \((v_n, p), (p, u_n) \in E(G)\) and the transitive property of \( G \), we obtain \((v_n, u_n) \in E(G)\). Since \( T_1, T_2 \) are asymptotically \( G \)-nonexpansive mappings, we get
\[
\| T_2^n u_n - u_n \|
\leq \| T_2^n u_n - T_2^n v_n \| + \| T_2^n v_n - T_1^n v_n \| + \| T_1^n v_n - T_1^n u_n \| + \| T_1^n u_n - u_n \|
\]
\[
\leq \| T_2^n u_n - T_2^n v_n \| + \| T_2^n v_n - T_1^n v_n \| + \| T_1^n v_n - T_1^n u_n \| + \| T_1^n u_n - u_n \|
\]
\[
\leq \| T_2^n u_n - T_2^n v_n \| + \| T_2^n v_n - T_1^n v_n \| + \| T_1^n v_n - T_1^n u_n \| + \| T_1^n u_n - u_n \|. \tag{2.21}
\]
\[ \leq \lambda_n \, \| v_n - u_n \| + \| T_i^n u_n - T_i^n v_n \| + \lambda_t \, \| v_n - u_n \| + \| T_i^n u_n - u_n \|. \]  \tag{2.30} \\

It follows from (2.25), (2.27), (2.29) and (2.30) that
\[ \lim_{n \to \infty} \| T_i^n u_n - u_n \| = 0. \]  \tag{2.31} \\

Now, by combining \((u_n, p), (p, u_{n+1}) \in E(G)\) and the transitive property of \(G\), we conclude that \((u_n, u_{n+1}) \in E(G)\). Then, for each \(i = 1, 2\), due to the fact that \(T_i\) is an asymptotically \(G\)-nonexpansive mapping, we have
\[ \| u_{n+1} - T_i^n u_{n+1} \| \leq \| u_{n+1} - u_n \| + \| u_n - T_i^n u_n \| + \| T_i^n u_n - T_i^n u_{n+1} \|. \]
\[ \leq \| u_{n+1} - u_n \| + \| u_n - T_i^n u_n \| + \lambda_n \, \| u_n - u_{n+1} \| \]
\[ = (1 + \lambda_n) \| u_{n+1} - u_n \| + \| u_n - T_i^n u_n \|. \]  \tag{2.32} \\

It follows from (2.14), (2.25), (2.31) and (2.32) that
\[ \lim_{n \to \infty} \| T_i^n u_n - T_i^n u_{n+1} \| = 0. \]  \tag{2.33} \\

Since \((p, u_{n+1}) \in E(G)\) for \(p \in F\) and \(T_i^n\) is edge-preserving, we have \((p, T_i^n u_{n+1}) \in E(G)\). By combining this with \((u_{n+1}, p) \in E(G)\) and using the transitive property of \(G\), we obtain \((u_{n+1}, T_i^n u_{n+1}) \in E(G)\). Since \(T_i\) is an asymptotically \(G\)-nonexpansive mapping, we have
\[ \| u_{n+1} - T_i^n u_{n+1} \| \leq \| u_{n+1} - u_n \| + \| u_n - T_i^n u_n \| + \| T_i^n u_n - T_i^n u_{n+1} \|. \]
\[ \leq \| u_{n+1} - u_n \| + \| u_n - T_i^n u_n \| + \lambda_n \, \| u_n - u_{n+1} \| \]. \]  \tag{2.34} \\

Taking the limit in (2.34) as \(n \to \infty\) and using (2.25), (2.31) and (2.33), we find that
\[ \lim_{n \to \infty} \| T_i^n u_n - u_n \| = 0. \]  \tag{2.35} \\

Therefore, by Lemma 1.5, (2.35), we find that \(T_i u = T_2 u = u\) and hence \(u \in F\).

**Step 6.** We show that \(u = q = P_q u_n\). Indeed, since \(u_n = P_q u_n\), we get
\[ \langle u_n - u_n, y - y \rangle \geq 0 \]  \tag{2.36} \\

for all \(y \in \Omega\). Let \(p \in F\). Since \(F \subset \Omega\), we have \(p \in \Omega\). Then, by choosing \(y = p\) in (2.36), we obtain \(\langle u_n - u_n, u_n - p \rangle \geq 0\). Taking the limit in this inequality as \(n \to \infty\) and using \(\lim_{n \to \infty} u_n = u\), we find that \(\langle u_n - u_n, u_n - p \rangle \geq 0\). This implies that \(u = P_p u_n\).

Since every \(G\)-nonexpansive mapping is an asymptotically \(G\)-nonexpansive mapping with the asymptotic coefficient \(\lambda_n = 1\) for all \(n \in \mathbb{N}\), from Theorem 2.2, we get the following corollary.

**Corollary 2.3.**

Let \(H\) be a real Hilbert space, \(H\) have property \((G), G = (V(G), E(G))\) be a directed transitive graph such that \(V(G) = H\), \(E(G)\) be coordinate-convex, \(T_1, T_2 : \Omega \to \Omega\) be two \(G\)-nonexpansive mappings such that \(F = F(T_1) \cap F(T_2) = \emptyset\), \(F(T_1) \times F(T_2) \subset E(G)\) for all \(i = 1, 2\), \{\(u_n\)\} be a sequence generated by (1.4) where \{\(a_n\), \(b_n\)\} are sequences in \([0, 1]\) such that \(0 < \lim_{n \to \infty} a_n \leq \lim_{n \to \infty} a_n < 1, 0 < \lim_{n \to \infty} b_n \leq \lim_{n \to \infty} b_n < 1; \) and \(\gamma_n \in [\alpha, \beta]\) for some
\[ \alpha, \beta \in \mathbb{R} \text{ such that } (u, p), (p, u), (z, p) \in E(G) \text{ for all } p \in F. \] Then the sequence \( \{u_n\} \) strongly converges to \( P_F u_1 \).

Finally, we give a numerical example to illustrate for the convergence of the proposed iteration processes. In addition, the example also shows that the convergence of the proposed iteration processes to common fixed points of given mappings faster than some previous iteration processes.

**Example 2.4.**

Let \( H = \mathbb{R}, \ G = (V(G), E(G)) \) be a directed graph defined by \( V(G) = H, \ E(G) = \{ (u, v) : u, v \in [1, +\infty) \text{ and } u = v \} \cup \{ (u, u) : u \in V(G) \}. \) Then \( E(G) \) is coordinate-convex and \( \{(u, u) : u \in V(G)\} \subset E(G). \) Define three mappings \( T_1, T_2, T_3 : H \rightarrow H \) by

\[ T_1u = \frac{1}{2} \sin^2(u - 1) + 1, \quad T_2u = T_3u = \frac{2u^2}{u^2 + 1} \text{ for all } u \in H. \]

Then, it is easy to check that \( T_1, T_2, T_3 \) are three asymptotically \( G \)-nonexpansive mappings with \( \lambda_n = 1 \) for all \( n \in \mathbb{N}. \) However, we see that \( T_2v - v = -\frac{v(v - 1)^2}{v^2 + 1} \leq 0 \) for all \( v \geq 0. \) This implies that \( 0 \leq T_2v \leq v \) for all \( v \geq 0. \) Therefore, \( 0 \leq T_2^2v = T_2(T_2v) \leq T_2v \leq v. \) By continuing this process, we get that \( 0 \leq T_2^n v \leq T_2^{n-1}v \leq ... T_2v \leq v \) for all \( v \geq 0 \) and \( n \in \mathbb{N}. \) By choosing \( u = 1 \) and \( v = 0.7, \) we obtain that \( 0 \leq T_2^n(0.7) \leq T_2(0.7) \leq 0.7 \) for all \( n \in \mathbb{N} \) and hence

\[ |T_2^n u - T_2^n v| = |T_2^n(1) - T_2^n(0.7)| = 1 - T_2^n(0.7) \geq 1 - T_2(0.7) = \frac{51}{149} > 0.3 = |u - v|. \]

This implies that the condition \( ||T_2^n u - T_2^n v|| \leq \lambda_n ||u - v|| \) is not satisfied for \( u = 1, v = 0.7 \) and for all \( \lambda_n \geq 1. \) Therefore, \( T_2 \) is not an asymptotically nonexpansive mapping. Thus, some convergence results for asymptotically nonexpansive mappings can be not applicable to \( T_2. \) We also have \( F = F(T_1) \cap F(T_2) \cap F(T_3) = \{1\} \neq \emptyset. \) Consider

\[ a_n = \frac{n + 2}{4n + 5}, b_n = \frac{n + 1}{3n + 7}, c_n = \frac{n + 2}{8n + 5} \quad \text{and} \quad \gamma_n = \frac{n + 1}{8n + 3} \quad \text{for all} \quad n \in \mathbb{N}. \]

By choosing \( u_3 = 3 \) and \( u_2 = 2.5. \) Then the numerical results of the iteration processes (1.1) – (1.5) are presented by the following table and figure.

<table>
<thead>
<tr>
<th>( n )</th>
<th>Iteration (1.1)</th>
<th>Iteration (1.2)</th>
<th>Iteration (1.3)</th>
<th>Iteration (1.4)</th>
<th>Iteration (1.5)</th>
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<td>3.000000</td>
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<td>3.000000</td>
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<tr>
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<tr>
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</tr>
<tr>
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</tr>
</tbody>
</table>

**Table 1. Numerical results of the iteration processes (1.1) – (1.5)**
Table 1 and Figure 1 show that for given mappings, the iteration processes (1.1) – (1.5) converge to 1. Furthermore, the convergence of the iteration process (1.5) to 1 is the fastest among other iteration processes. For the iteration processes for two \( G \)-nonexpansive mappings, the convergence of the iteration process (1.4) to 1 is faster than the iteration process (1.3). For the iteration processes for asymptotically \( G \)-nonexpansive mappings, the convergence of the iteration process (1.5) to 1 is faster than the iteration process (1.1) and (1.2).

\[ \begin{array}{cccc}
35 & 1.0000166 & 1.0000111 & 1.0000001 & 1. & 1. \\
36 & 1.0000124 & 1.0000083 & 1. & 1. & 1. \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
53 & 1.0000001 & 1.0000001 & 1. & 1. & 1. \\
54 & 1.0000001 & 1. & 1. & 1. & 1. \\
55 & 1. & 1. & 1. & 1. & 1. \\
\end{array} \]

Figure 1. Comparison of the convergence of iteration processes (1.1) – (1.5).

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TÔM TÁT

Từ khóa: ánh xạ G-không giãn tiệm cần; không gian Hilbert với độ thị; dãy lập lại giúp có yếu tố quản tính