LOWER SEMICONTINUITY OF THE SOLUTION SETS OF PARAMETRIC GENERALIZED QUASIEQUILIBRIUM PROBLEMS

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ABSTRACT

In this paper we establish sufficient conditions for the solution sets of parametric generalized quasiequilibrium problems with the stability properties such as lower semicontinuity and Hausdorff lower semicontinuity.

Keyword: parametric generalized quasiequilibrium problems, lower semicontinuity, Hausdorff lower semicontinuity.

TÓM TẮT

Tính chất nửa liên tục dưới của các tập nghiệm của các bài toán tựa cân bằng tổng quát phụ thuộc tham số

Trong bài báo này, chúng tôi thiết lập điều kiện đủ cho các tập nghiệm của các bài toán tựa cân bằng tổng quát phụ thuộc tham số có các tính chất ổn định như: tính nửa liên tục dưới và tính nửa liên tục dưới Hausdorff.

Từ khóa: các bài toán tựa cân bằng tổng quát phụ thuộc tham số, tính nửa liên tục dưới, tính nửa liên tục dưới Hausdorff.

1. Introduction and Preliminaries

Let X, Y, Λ, Γ, M be a Hausdorff topological spaces, let Z be a Hausdorff topological vector space, $A \subseteq X$ and $B \subseteq Y$ be a nonempty sets. Let $K_1 : A \times \Lambda \to 2^A$, $K_2 : A \times \Lambda \to 2^A$, $T : A \times A \times \Gamma \to 2^B$, $C : A \times \Lambda \to 2^B$ and $F : A \times B \times A \times M \to 2^Z$ be multifunctions with C is a proper solid convex cone values and closed.

For the sake of simplicity, we adopt the following notations. Letters w, m and s are used for a weak, middle and strong, respectively, kinds of considered problems. For ubsets U and V under consideration we adopt the notations.

(u,v) w $U \times V$ means $\forall u \in U, \exists v \in V$,

(u,v) m $U \times V$ means $\exists v \in V, \forall u \in U$,

(u,v) s $U \times V$ means $\forall u \in U, \forall v \in V$,

$\rho_1(U,V)$ means $U \cap V \neq \emptyset$,

$\rho_2(U,V)$ means $U \subseteq V$,

(u,v) $\overline{w} U \times V$ means $\exists u \in U, \forall v \in V$ and similarly for $\overline{m}, \overline{s}$,

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$\overline{\rho}_1(U,V)$ means $U \cap V = \varnothing$ and similarly for $\overline{\rho}_2$.

Let $\alpha \in \{\text{w, m, s}\}$, $\overline{\alpha} \in \{\overline{w}, \overline{m}, \overline{s}\}$, $\rho \in \{\rho_1, \rho_2\}$ and $\overline{\rho} \in \{\overline{\rho_1}, \overline{\rho_2}\}$. We consider the following parametric generalized quasiequilibrium problems.

(QEP $_{\alpha\rho}$): Find $\overline{x} \in K_1(\overline{x}, \lambda)$ such that $(y,t)\alpha K_2(\overline{x}, \lambda) \times T(\overline{x}, y, \gamma)$ satisfying

$\rho(F(\overline{x}, t, y, \mu); C(\overline{x}, \lambda)).$

We consider also the following problem (QEP$^*_{\alpha\rho}$) as an auxiliary problem to (QEP $_{\alpha\rho}$):

(QEP$^*_{\alpha\rho}$): Find $\overline{x} \in K_1(\overline{x}, \lambda)$ such that $(y,t)\alpha K_2(\overline{x}, \lambda) \times T(\overline{x}, y, \gamma)$ satisfying

$\rho(F(\overline{x}, t, y, \mu); \operatorname{int} C(\overline{x}, \lambda)).$

For each $\lambda \in \Lambda, \gamma \in \Gamma, \mu \in M$, we let $E(\lambda) := \{x \in A \mid x \in K_1(x, \lambda)\}$ and let $\Sigma_{\alpha\rho}, \tilde{\Sigma}_{\alpha\rho} : \Lambda \times \Gamma \times M \to 2^A$ be a set-valued mapping such that $\Sigma_{\alpha\rho}(\lambda, \gamma, \mu)$ and $\tilde{\Sigma}_{\alpha\rho}(\lambda, \gamma, \mu)$ are the solution sets of (QEP $_{\alpha\rho}$) and (QEP$^*_{\alpha\rho}$), respectively, i.e.,

$$\Sigma_{\alpha\rho}(\lambda, \gamma, \mu) = \{\overline{x} \in E(\lambda) \mid (y,t)\alpha K_2(\overline{x}, \lambda) \times T(\overline{x}, y, \gamma) : \rho(F(\overline{x}, t, y, \mu); C(\overline{x}, \lambda))\},$$

$$\tilde{\Sigma}_{\alpha\rho}(\lambda, \gamma, \mu) = \{\overline{x} \in E(\lambda) \mid (y,t)\alpha K_2(\overline{x}, \lambda) \times T(\overline{x}, y, \gamma) : \rho(F(\overline{x}, t, y, \mu); \operatorname{int} C(\overline{x}, \lambda))\}.$$

Clearly $\tilde{\Sigma}_{\alpha\rho}(\lambda, \gamma, \mu) \subseteq \Sigma_{\alpha\rho}(\lambda, \gamma, \mu)$. Throughout the paper we assume that $\Sigma_{\alpha\rho}(\lambda, \gamma, \mu) \neq \varnothing$ and $\tilde{\Sigma}_{\alpha\rho}(\lambda, \gamma, \mu) \neq \varnothing$ for each (λ, γ, μ) in the neighborhood of $(\lambda_0, \gamma_0, \mu_0) \in \Lambda \times \Gamma \times M$.

By the definition, the following relations are clear:

$\Sigma_{s\rho} \subseteq \Sigma_{m\rho} \subseteq \Sigma_{w\rho}$ and $\tilde{\Sigma}_{s\rho} \subseteq \tilde{\Sigma}_{m\rho} \subseteq \tilde{\Sigma}_{w\rho}$.

The parametric generalized quasiequilibrium problems is more general than many following problems.

(a) If $T(x, y, \gamma) = \{x\}, \Lambda = \Gamma = M, A = B, X = Y, K_1 = K_2 = K, \rho = \overline{\rho_2}, \rho = \overline{\rho_1}$ and replace $C(x, \lambda)$ by $-\operatorname{int} C(x, \lambda)$. Then, (QEP $_{\alpha\overline{\rho_2}}$) and (QEP $_{\alpha\overline{\rho_1}}$) becomes to (PGQVEP) and (PEQVEP), respectively, in Kimura-Yao [7].

(PGQVEP): Find $\overline{x} \in K(\overline{x}, \lambda)$ such that

$F(\overline{x}, y, \lambda) \not\subset -\operatorname{int} C(\overline{x}, \lambda)),$ for all $y \in K(x, \lambda)$.

and

(PEQVEP): Find $\overline{x} \in K(\overline{x}, \lambda)$ such that

$F(\overline{x}, y, \lambda) \cap (-\operatorname{int} C(\overline{x}, \lambda)) = \varnothing,$ for all $y \in K(x, \lambda)$.

(b) If $T(x,y,\gamma)=\{x\}, \Lambda=\Gamma, A=B, X=Y, K_1=clK, K_2=K, \rho=\rho_1, \rho=\rho_2$ and replace $C(x,\lambda)$ by $Z\setminus -\operatorname{int}C$ with $C\subseteq Z$ be closed and $\operatorname{int}C\neq\emptyset$. Then, (QEP $_{\alpha\rho_1}$) and (QEP $_{\alpha\rho_2}$) becomes to (QEP) and (SQEP), respectively, in Anh - Khanh [1].

(QEP): Find $\bar{x}\in clK(\bar{x},\lambda)$ such that

$F(\bar{x},y,\lambda)\cap(Z\setminus -\operatorname{int}C)\neq\emptyset,$ for all $y\in K(x,\lambda).$

and

(SQEP): Find $\bar{x}\in K(\bar{x},\lambda)$ such that

$F(\bar{x},y,\lambda)\subseteq Z\setminus -\operatorname{int}C,$ for all $y\in K(x,\lambda).$

(c) If $T(x,y,\gamma)=\{x\}, \Lambda=\Gamma=M, A=B, X=Y, K_1=K_2=K, \rho=\overline{\rho_2}$ and replace $C(x,\lambda)$ by $-\operatorname{int}C(x,\lambda)$, replace F by f be a vector function. Then, (QEP $_{\alpha\overline{\rho_2}}$) becomes to (PVQEP) in Kimura-Yao [6].

(PQVEP): Find $\bar{x}\in K(\bar{x},\lambda)$ such that

$f(\bar{x},y,\lambda)\notin -\operatorname{int}C(\bar{x},\lambda)),$ for all $y\in K(x,\lambda).$

Note that generalized quasiequilibrium problems encompass many optimization-related models like vector minimization, variation inequalities, Nash equilibrium, fixed point and coincidence-point problems, complementary problems, minimum inequalities, etc. Stability properties of solutions have been investigated even in models for vector quasiequilibrium problems [1, 2, 3, 6, 7, 8], variation problems [4, 5, 9, 10] and the references therein.

In this paper we establish sufficient conditions for the solution sets $\Sigma_{\alpha\rho}$ to have the stability properties such as the lower semicontinuity and the Hausdorff lower semicontinuity with respect to parameter λ,γ,μ under relaxed assumptions about generalized convexity of the map F.

The structure of our paper is as follows. In the remaining part of this section, we recall definitions for later uses. Section 2 is devoted to the lower semicontinuity and the Hausdorff lower semicontinuity of solution sets of problems (QEP $_{\alpha\rho}$).

Now we recall some notions. Let X and Z be as above and $G: X\to 2^Z$ be a multifunction. G is said to be lower semicontinuous (lsc) at x_0 if $G(x_0)\cap U\neq\emptyset$ for some open set $U\subseteq Z$ implies the existence of a neighborhood N of x_0 such that, for all $x\in N, G(x)\cap U\neq\emptyset$. An equivalent formulation is that: G is lsc at x_0 if $\forall x_\alpha\to x_0$, $\forall z_0\in G(x_0), \exists z_\alpha\in G(x_\alpha), z_\alpha\to z_0$. G is called upper semicontinuous (usc) at x_0 if for each open set $U\supseteq G(x_0)$, there is a neighborhood N of x_0 such that $U\supseteq G(N)$. Q is said to be Hausdorff upper semicontinuous (H-usc in short; Hausdorff lower semicontinuous, H-lsc, respectively) at x_0 if for each neighborhood B of the origin in Z, there exists a neighborhood N of x_0 such that, $Q(x)\subseteq Q(x_0)+B, \forall x\in N$

($Q(x_0) \subseteq Q(x) + B, \forall x \in N$). G is said to be continuous at x_0 if it is both lsc and usc at x_0 and to be H-continuous at x_0 if it is both H-lsc and H-usc at x_0. G is called closed at x_0 if for each net $\{(x_\alpha, z_\alpha)\} \subseteq \text{graph}G := \{(x,z) \mid z \in G(x)\}, (x_\alpha, z_\alpha) \to (x_0, z_0)$, z_0 must belong to $G(x_0)$. The closeness is closely related to the upper (and Hausdorff upper) semicontinuity. We say that G satisfies a certain property in a subset $A \subseteq X$ if G satisfies it at every points of A. If $A = X$ we omit ``in X" in the statement.

Let A and Z be as above and $G : A \to 2^Z$ be a multifunction.

(i) If G is usc at x_0 then G is H-usc at x_0. Conversely if G is H-usc at x_0 and if $G(x_0)$ compact, then G usc at x_0;

(ii) If G is H-lsc at x_0 then G is lsc. The converse is true if $G(x_0)$ is compact;

(iii) If G has compact values, then G is usc at x_0 if and only if, for each net $\{x_\alpha\} \subseteq A$ which converges to x_0 and for each net $\{y_\alpha\} \subseteq G(x_\alpha)$, there are $y \in G(x)$ and a subnet $\{y_\beta\}$ of $\{y_\alpha\}$ such that $y_\beta \to y$.

Definition. *(See [1], [11])* Let X and Z be as above. Suppose that A is a nonempty convex set of X and that $G : X \to 2^Z$ be a multifunction.

(i) G is said to be *convex* in A if for each $x_1, x_2 \in A$ and $t \in [0,1]$

$$G(tx_1 + (1-t)x_2) \supset tG(x_1) + (1-t)G(x_2)$$

(ii) G is said to be *concave* A if for each $x_1, x_2 \in A$ and $t \in [0,1]$

$$G(tx_1 + (1-t)x_2) \subset tG(x_1) + (1-t)G(x_2)$$

2. Main results

In this section, we discuss the lower semicontinuity and the Hausdorff lower semicontinuity of solution sets for parametric generalized quasiequilibrium problems (QEP$_{\alpha\rho}$).

Definition 2.1

Let A and Z be as above and $C : A \to 2^Z$ with a proper solid convex cone values. Suppose $G : A \to 2^Z$. We say that G *is generalized* C*-concave in* A if for each $x_1, x_2 \in A$, $\rho(G(x_1), C(x_1))$ and $\rho(G(x_2), \text{int}\, C(x_2))$ imply

$$\rho(G(tx_1 + (1-t)x_2), \text{int}\, C(tx_1 + (1-t)x_2)), \text{ for all } t \in (0,1).$$

Theorem 2.2

Assume for problem (QEP$_{\alpha\rho}$) that

(i) E is lsc at λ_0, K_2 is usc and compact-valued in $K_1(A,\Lambda) \times \{\lambda_0\}$ and $E(\lambda_0)$ is convex;

(ii) in $K_1(A,\Lambda)\times K_2(K_1(A,\Lambda),\Lambda)\times\{\gamma_0\}$*, T is usc and compact-valued if* $\alpha = s$*, and lsc if* $\alpha = w$ *(or* $\alpha = m$*);*

(iii) $\forall t\in T(K_1(A,\Lambda)\times K_2(K_1(A,\Lambda),\Lambda),\Gamma), \forall\mu_0\in M, \forall\lambda_0\in\Lambda$*,* $K_2(.,\lambda_0)$ *is concave in* $K_1(A,\Lambda)$ *and* $F(.,t,.,\mu_0)$ *is generalized* $C(.,\lambda_0)$*-concave in* $K_1(A,\Lambda)\times K_2(K_1(A,\Lambda),\Lambda)$*;*

(iv) the set $\{(x,t,y,\mu,\lambda)\in K_1(A,\Lambda)\times T(K_1(A,\Lambda),K_2(K_1(A,\Lambda),\Lambda),\Gamma)\times K_2(K_1(A,\Lambda),\Lambda)\times \{\mu_0\}\times\{\lambda_0\} : \bar{\rho}(F(x,t,y,\mu);C(x,\lambda))\}$ *is closed.*

Then $\Sigma_{\alpha\rho}$ *is lower semicontinuous at* $(\lambda_0,\gamma_0,\mu_0)$*.*

Proof.

Since $\alpha=\{w,m,s\}$ and $\rho=\{\rho_1,\rho_2\}$, we have in fact six cases. However, the proof techniques are similar. We consider only the cases $\alpha=s, \rho=\rho_2$. We prove that $\tilde{\Sigma}_{s\rho_2}$ is lower semicontinuous at $(\lambda_0,\gamma_0,\mu_0)$. Suppose to the contrary that $\tilde{\Sigma}_{s\rho_2}$ is not lsc at $(\lambda_0,\gamma_0,\mu_0)$, i.e., $\exists x_0\in\tilde{\Sigma}_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)$, $\exists(\lambda_n,\gamma_n,\mu_n)\to(\lambda_0,\gamma_0,\mu_0)$, $\forall x_n\in\tilde{\Sigma}_{s\rho_2}(\lambda_n,\gamma_n,\mu_n)$, $x_n\not\to x_0$. Since E is lsc at λ_0, there is a net $x'_n\in E(\lambda_n)$, $x'_n\to x_0$. By the above contradiction assumption, there must be a subnet x'_m of x'_n such that, $\forall m$, $x'_m\notin\tilde{\Sigma}_{s\rho_2}(\lambda_m,\gamma_m,\mu_m)$, i.e., $\exists y_m\in K_2(x'_m,\lambda_m)$, $\exists t_m\in T(x'_m,y_m,\gamma_m)$ such that

$$F(x'_m,t_m,y_m,\mu_m)\not\subseteq \operatorname{int}C(x'_m,\lambda_m). \qquad (2.1)$$

As K_2 is usc at (x_0,λ_0) and $K_2(x_0,\lambda_0)$ is compact, one has $y_0\in K_2(x_0,\lambda_0)$ such that $y_m\to y_0$ (taking a subnet if necessary). By the lower semicontinuity of T at (x_0,y_0,γ_0),

one has $t_m\in T(x_m,y_m,\gamma_m)$ such that $t_m\to t_0$.

Since $(x'_m,t_m,y_m,\lambda_m,\gamma_m,\mu_m)\to (x_0,t_0,y_0,\lambda_0,\gamma_0,\mu_0)$ and by condition (iv) and (2.1) yields that

$$F(x_0,t_0,y_0,\mu_0)\not\subseteq \operatorname{int}C(x_0,\lambda_0),$$

which is impossible since $x_0\in\tilde{\Sigma}_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)$. Therefore, $\tilde{\Sigma}_{s\rho_2}$ is lsc at $(\lambda_0,\gamma_0,\mu_0)$.

Now we check that

$$\Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)\subseteq \operatorname{cl}(\tilde{\Sigma}_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)).$$

Indeed, let $x_1\in\Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)$, $x_2\in\tilde{\Sigma}_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)$ and $x_\alpha=(1-t)x_1+tx_2, t\in(0,1)$. By the convexity of E, we have $x_\alpha\in E(\lambda_0)$. By the generalized $C(.,\lambda_0)$-concavity of $F(.,t,y,\mu_0)$, we have

$$F(x_\alpha,t,y,\mu_0)\subseteq \operatorname{int}C(x_\alpha,\lambda_0),$$

and since $K_2(.,\lambda_0)$ is concave, one implies that for each $y_\alpha \in K_2(x_\alpha,\lambda_0)$, there exist $y_1 \in K_2(x_1,\lambda_0)$ and $y_2 \in K_2(x_2,\lambda_0)$ such that $y_\alpha = ty_1 + (1-t)y_2$. By the generalized $C(.,\lambda_0)$-concavity of $F(.,t,.,\mu_0)$, we have

$$F(x_\alpha, t, y_\alpha, \mu_0) \subseteq \operatorname{int} C(x_\alpha,\lambda_0),$$

i.e., $x_\alpha \in \tilde{\Sigma}_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)$. Hence $\Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0) \subseteq cl(\tilde{\Sigma}_{s\rho_2}(\lambda_0,\gamma_0,\mu_0))$. By the lower semicontinuity of $\tilde{\Sigma}_{s\rho_2}$ at $(\lambda_0,\gamma_0,\mu_0)$, we have

$$\Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0) \subseteq cl(\tilde{\Sigma}_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)) \subseteq \liminf \tilde{\Sigma}_{s\rho_2}(\lambda_n,\gamma_n,\mu_n) \subseteq \liminf \Sigma_{s\rho_2}(\lambda_n,\gamma_n,\mu_n),$$

i.e., $\Sigma_{s\rho_2}$ is lower semicontinuous at $(\lambda_0,\gamma_0,\mu_0)$. □

The following example shows that the lower semicontinuity of E is essential.

Example 2.3

Let $A = B = X = Y = Z = \mathbb{R}$, $\Lambda = \Gamma = M = [0,1]$, $\lambda_0 = 0, C(x,\lambda) = [0,+\infty)$ and let $F(x,t,y,\lambda) = 2^\lambda, T(x,y,\lambda) = \{x\}, K_2(x,\lambda) = [0,1]$

and

$$K_1(x,\lambda) = \begin{cases} [-1,1] & \text{if } \lambda = 0, \\ [-1-\lambda, 0] & \textit{otherwise.} \end{cases}$$

We have $E(0) = [-1,1]$, $E(\lambda) = [-\lambda-1, 0], \forall \lambda \in (0,1]$. Hence K_2 is usc and the condition (ii), (iii) and (iv) of Theorem 2.2 is easily seen to be fulfilled. But $\Sigma_{\alpha\rho}$ is not upper semicontinuous at $\lambda_0 = 0$. The reason is that E is not lower semicontinuous. In fact $\Sigma_{\alpha\rho}(0,0,0) = [-1,1]$ and $\Sigma_{\alpha\rho}(\lambda,\gamma,\mu) = [-\lambda-1,0], \forall \lambda \in (0,1]$.

The following example shows that in this the special case, assumption (iv) of Theorem 2.2 may be satisfied even in cases, but both assumption (ii_1) and (iii_1) of Theorem 2.1 in Anh-Khanh [1] are not fulfilled.

Example 2.4

Let $A, B, X, Y, Z, T, \Lambda, \Gamma, M, \lambda_0, C$ as in Example 2.3, and let $K_1(x,\lambda) = K_2(x,\lambda) = [0,1]$ and

$$K_1(x,\lambda) = \begin{cases} [-4,0] & \text{if } \lambda = 0, \\ [-1-\lambda, 0] & \textit{otherwise.} \end{cases}$$

We shows that the assumptions (i), (ii) and (iii) of Theorem 2.2 satisfied and

$\Sigma_{\alpha\rho}(\lambda,\gamma,\mu) = [0,1], \forall \lambda \in [0,1]$. But both assumption ($ii_1$) and ($iii_1$) of Theorem 2.1 in Anh-Khanh [1] are not fulfilled.

The following example shows that in this the special case, assumption of Theorem2.2 may be satisfied even in cases, but Theorem 2.1 and Theorem 2.3 in Anh-Khanh [1] are not fulfilled.

Example 2.5

Let $A, B, X, Y, T, \Lambda, \Gamma, M, \lambda_0, C$ as in Example 2.4, and let $K_1(x,\lambda) = K_2(x,\lambda) =$ $[0, \frac{\lambda}{2}]$ and

$$K_1(x,\lambda) = \begin{cases} [0,1] & \text{if } \lambda = 0, \\ [2,4] & otherwise. \end{cases}$$

We shows that the assumptions (i), (ii) and (iii) of Theorem 2.2 satisfied and $\Sigma_{\alpha\rho}(\lambda,\gamma,\mu)) =$ $[0, \frac{\lambda}{2}], \forall \lambda \in [0,1]$. Theorem 2.1 and Theorem 2.3 in Anh-Khanh [1] are not fulfilled. The reason is that F is neither usc nor lsc at $(x, y, 0)$.

Remark 2.6

In special cases, as in Section 1 (a) and (c). Then, Theorem 2.2 reduces to Theorem 5.1 in Kimura-Yao [7, 6]. However, the proof of the theorem 5.1 is in a different way. Its assumption (i) - (v) of Theorem 5.1 coincides with (i) of Theorem 2.2 and assumption (vi), (vii) coincides with (iii), (iv) of Theorem 2.2 Theorem 2.2 slightly improves Theorem 5.1 in Kimura-Yao [7, 6], since no convexity of the values of E is imposed.

The following example shows that the convexity and lower semicontinuity of K is essential.

Example 2.7

Let $A, X, Y, Z, C, \Lambda, M, \Gamma, \lambda_0$ as in Example 2.5 and let

$$K_1(x,\lambda) = \begin{cases} \{-1,0,1\} & \text{if } \lambda = 0, \\ \{0,1\} & otherwise. \end{cases}$$

Then, we shows that K_2 is usc and has compact-valued $K_1(X, A) \times \{\lambda_0\}$ and assumption (ii), (iii) and (iv) of Theorem 2.2 are fulfilled. But $\Sigma_{\alpha\rho}(\lambda,\gamma,\mu))$ is not lsc at $(0,0,0)$. The reason is that E is not lsc at $\lambda_0 = 0$ and $E(0)$ is also not convex. Indeed, let $x_1 = -1, x_2 = 0 \in E(0)$ and $t = \frac{1}{2} \in (0,1)$ but $tx_1 + (1-t)x_2 \notin E(0)$.

In fact, $\Sigma_{\alpha\rho}(0,0,0) = \{-1,0,1\}$ and $\Sigma_{\alpha\rho}(\lambda,\gamma,\mu) = \{0,1\}, \forall \lambda \in (0,1]$.

The following example shows that the concavity of $F(.,t.,\mu_0)$ is essential.

Example 2.8

Let $A, X, Y, Z, C, \Lambda, M, \Gamma, \lambda_0$ as in Example 2.6 and let $K_1(x,\lambda) = K_2(x,\lambda)$ $= [\lambda, \lambda+3]$ and $F(x,t,y,\mu) = F(x,y,\lambda) = x^2 - (1+\lambda)x$. We show that $K_2(.,\lambda_0)$ is concave and the assumptions (i), (ii), (iv) of Theorem 2.2. are satisfied. But $\Sigma_{\alpha\rho}$ is not lsc at $(0,0,0)$. The reason is that the concavity of F is violated. Indeed, taking $x_1 = 0, x_2 = \dfrac{3}{2} \in E(0) = [0,3]$, then for all $y \in K_2(A,0) = [0,3]$, we have $F(x_1, y, 0) = 0, F(x_2, y, 0) = 3/4$, but $F(\frac{1}{2}x_1 + \frac{1}{2}x_2, y, 0) = -\frac{3}{16} \notin (0,+\infty)$.

Theorem 2.9

Impose the assumption of Theorem 2.2 and the following additional conditions:

(v) K_2 is lsc in $K_1(A,\Lambda)\times\{\lambda_0\}$ and $E(\lambda_0)$ is compact;

(vi) the set $\{(x,t,y) \in K_1(A,\Lambda)\times T(K_1(A,\Lambda), K_2(K_1(A,\Lambda),\Lambda),\Gamma)\times K_2(K_1(A,\Lambda),\Lambda):$ $\rho(F(x,t,y,\mu_0); C(x,\lambda_0))\}$ is closed.

Then $\Sigma_{\alpha\rho}$ is Hausdorff lower semicontinuous at $(\lambda_0, \gamma_0, \mu_0)$.

Proof.

We consider only for the cases: $\alpha = s, \rho = \rho_2$. We first prove that $\Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)$ is closed. Indeed, we let $x_n \in \Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)$ such that $x_n \to x_0$. If $x_0 \notin \Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)$,

$\exists y_0 \in K_2(x_0,\lambda_0), \exists t_0 \in T(x_0, y_0, \gamma_0)$ such that

$$F(x_0,t_0,y_0,\mu_0) \not\subseteq C(x_0,\lambda_0). \tag{2.2}$$

By the lower semicontinuity of $K_2(.,\lambda_0)$ at x_0, one has $y_n \in K_2(x_n,\lambda_0)$ such that $y_n \to y_0$. Since $x_n \in \Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)$, $\forall t_n \in T(x_n, y_n, \gamma_0)$ such that

$$F(x_n,t_n,y_n,\mu_0) \subseteq C(x_n,\lambda_0). \tag{2.3}$$

By the condition (vi), we see a contradiction between (2.2) and (2.3). Therefore, $\Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)$ is closed.

On the other hand, since $\Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0) \subseteq E(\lambda_0)$ is compact by $E(\lambda_0)$ compact. Since $\Sigma_{s\rho_2}$ is lower semicontinuous at $(\lambda_0,\gamma_0,\mu_0)$ and $\Sigma_{s\rho_2}(\lambda_0,\gamma_0,\mu_0)$ compact. Hence $\Sigma_{s\rho_2}$ is Hausdorff lower semicontinuous at $(\lambda_0,\gamma_0,\mu_0)$. So we complete the proof.

The following example shows that the assumed compactness in (v) is essential.

Example 2.10

Let $X = Y = A = B = \mathbb{R}^2, Z = \mathbb{R}, \Lambda = M = \Gamma = [0,1], C(x,\lambda) = \mathbb{R}_+, \lambda_0 = 0$, and for $x = (x-1, x_2) \in \mathbb{R}^2, K_1(x,\lambda) = K_1(x,\lambda) = \{(x_1, \lambda x_1)\}$ and $F(x,t,y,\mu) = 1+\lambda$. We shows that the assumptions of Theorem 2.8 are satisfied, but the compactness of $E(\lambda_0)$ is not satisfied. Direct computations give $\Sigma_{\alpha\rho}(\lambda,\gamma,\mu)=\{(x_1,x_2)\in\mathbb{R}^2 \mid x_2=\lambda x_1\}$ and then $\Sigma_{\alpha\rho}$ is not Hausdorff lower semicontinuous at $(0,0,0)$ (although $\Sigma_{\alpha\rho}$ is lsc at (0,0,0)).

REFERENCES

1. Anh L. Q., Khanh P. Q. (2004), "Semicontinuity of the solution sets of parametric multivalued vector quasiequilibrium problems", *J. Math. Anal. Appl*., **294**, pp. 699-711.
2. Bianchi M., Pini R. (2003), "A note on stability for parametric equilibrium problems". *Oper. Res. Lett*., **31**, pp. 445-450.
3. Bianchi M., Pini R. (2006), "Sensitivity for parametric vector equilibria", *Optimization*., **55**, pp. 221-230.
4. Khanh P. Q., Luu L. M. (2005), "Upper semicontinuity of the solution set of parametric multivalued vector quasivariational inequalities and applications", *J. Glob.Optim.,* **32**, pp. 551-568.
5. Khanh P. Q., Luu L. M. (2007), "Lower and upper semicontinuity of the solution sets and approximate solution sets to parametric multivalued quasivariational inequalities", *J. Optim. Theory Appl*., **133**, pp. 329-339.
6. Kimura K., Yao J. C. (2008), "Sensitivity analysis of solution mappings of parametric vector quasiequilibrium problems", *J. Glob. Optim*., **41** pp. 187-202.
7. Kimura K., Yao J. C. (2008), "Sensitivity analysis of solution mappings of parametric generalized quasi vector equilibrium problems", *Taiwanese J. Math*., **9**, pp. 2233-2268.
8. Kimura K., Yao J. C. (2008), "Semicontinuity of Solution Mappings of parametric Generalized Vector Equilibrium Problems", *J. Optim. Theory Appl*., **138**, pp. 429–443.
9. Lalitha C. S., Bhatia Guneet. (2011), "Stability of parametric quasivariational inequality of the Minty type", *J. Optim. Theory Appl*., **148**, pp. 281-300.
10. Li S. J., Chen G. Y., Teo K. L. (2002), "On the stability of generalized vector quasivariational inequality problems", *J. Optim. Theory Appl*., 113, pp. 283-295.
11. Luc D. T. (1989), *Theory of Vector Optimization*: Lecture Notes in Economics and Mathematical Systems, Springer-Verlag Berlin Heidelberg.

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