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Research Article

WEIGHTED NORM INEQUALITIES OF GENERALIZED WEIGHTED HARDY-CESÀRO OPERATORS AND COMMUTATORS WITH SYMBOLS IN CMO SPACES ON GENERALIZED WEIGHTED MORREY SPACES

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ABSTRACT

In this work, our main aim is to study the boundedness of the weighted Hardy-Cesàro operators and commutators on generalized weighted Morrey spaces $M_{p,\varphi}(\omega)$. *We establish certain sufficient conditions which imply the boundedness of the weighted Hardy-Cesàro operators and their commutators with symbols in CMO spaces on generalized weighted Morrey spaces* $M_{p,\varphi}(\omega)$.

Keywords: weighted Hardy-Cesàro operator; commutator; generalized weighted Morrey space; CMO space

1. Introduction

Consider the classical Hardy operator U defined by

$$Uf(x) = \frac{1}{x}\int_0^x f(t)dt, x \neq 0$$

for $f \in L^1_{\mathrm{loc}}(\mathbb{R})$. A celebrated Hardy integral inequality, see (Guliyev, 2012), can be formulated as

$$\| Uf \|_{L^p(\mathbb{R})} \leq \frac{p}{p-1} \| f \|_{L^p(\mathbb{R})},$$

where $1 < p < \infty$, in which the constant $\frac{p}{p-1}$ is known as the best constant. The Hardy integral inequality and its generalizations then have been studied extensively since they

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play an important role in various branches of analysis such as approximation theory, differential equations, the theory of function spaces.

The generalized Hardy operator was first introduced in 1984 by C. Carton-Lebrun and M. Fosset (Carton-Lebrun, & Fosset, 1984), in which the authors defined the weighted Hardy operator U_ψ as follows. Let $\psi:[0,1]\to[0,\infty)$ be a measurable function, and let f be a measurable complex-valued function on $\mathbb{R}^n$. Then the weighted Hardy operator U_ψ is defined by

$$U_\psi f(x)=\int_0^1 f(tx)\psi(t)dt,\ x\in\mathbb{R}^n.$$

It was in the work mentioned above that C. Carton-Lebrun and M. Fosset showed that U_ψ is bounded on $BMO(\mathbb{R}^n)$.

Then in 2001, J. Xiao proved in (Xiao, 2001) that U_ψ is bounded on $L^p(\mathbb{R}^n)$ if and only if

$$\mathcal{B}:=\int_0^1 t^{-n/p}\psi(t)dt<\infty$$

and that the corresponding operator norm is exactly $\mathcal{B}$. Also, J. Xiao obtained $BMO(\mathbb{R}^n)-$bounds of U_ψ which sharpened the main result (Carton-Lebrun, & Fosset, 1984).

Recently, Z. W. Fu, Z. G. Liu and S. Z. Lu in 2009 presented a necessary and sufficient condition on the weight function ψ which characterizes the boundedness of the commutators of weighted Hardy operators U_ψ on $L^p(\mathbb{R}^n)$, $1<p<\infty$, with symbols in $BMO(\mathbb{R}^n)$ (Fu, Liu, & Lu, 2009).

In addition, the topic of boundedness of U_ψ and its commutator has been investigated extensively on classical Morrey spaces, Campanato spaces, Triebel-Lizorkin-type spaces by a number of authors (see (Fu, & Lu, 2010), (Kuang, 2010), (Tang, & Zhai, 2010) and (Tang, & Zhou, 2012)).

Inspired by the above papers, in this work we consider the generalized weighted Hardy-Cesàro operator and its commutator, which are defined as follows.

Definition 1.1.

Let $\psi:[0,1]\to[0,\infty)$ and $s:[0,1]\to\mathbb{R}$ be measurable functions. Then the generalized weighted Hardy-Cesàro operator $U_{\psi,s}$ is defined by

$$U_{\psi,s}f(x)=\int_0^1 f\left(s(t)x\right)\psi(t)dt,$$

for measurable complex-valued functions f on $\mathbb{R}^n$.

Definition 1.2.

Let b be a locally integrable function on $\mathbb{R}^n$. The commutator of b and the operator $U_{\psi,s}$ is defined by

$$U_{\psi,s}^b f = bU_{\psi,s}(f) - U_{\psi,s}(bf).$$

Our primary goal in this paper is to study weighted norm inequalities for the generalized weighted Hardy-Cesàro operator $U_{\psi,s}$ and its commutator $U_{\psi,s}^b$, with symbols b being CMO functions, on generalized weighted Morrey spaces $M_{p,\varphi}(\omega)$ which are introduced by V. S. Guliyev in (Guliyev, 2012) as follows.

Definition 1.3.

Let $0 < p < \infty$, φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$ and let ω be a weight function on $\mathbb{R}^n$. We denote by $M_{p,\varphi}(\omega)$ the generalized weighted Morrey space which is defined by

$$M_{p,\varphi}(\omega) = \left\{ f \in L_{loc}^{p,\omega}(\mathbb{R}^n) : \|f\|_{M_{p,\varphi}(\omega)} = \sup_{x\in\mathbb{R}^n, r>0} [\varphi(x,r)]^{-1}[\omega(B(x,r))]^{-\frac{1}{p}} \|f\|_{L^{p,\omega}(B(x,r))} < \infty \right\},$$

and by $M_{p,\varphi}^{cen}(\omega)$ the generalized weighted central Morrey space which is defined by

$$M_{p,\varphi}^{cen}(\omega) = \left\{ f \in L_{loc}^{p,\omega}(\mathbb{R}^n) : \|f\|_{M_{p,\varphi}^{cen}(\omega)} = \sup_{r>0} [\varphi(0,r)]^{-1}[\omega(B(0,r))]^{-\frac{1}{p}} \|f\|_{L^{p,\omega}(B(0,r))} < \infty \right\}.$$

Precisely speaking, we present certain sufficient conditions imposed on the functions s, ψ, φ and ω which guarantee the boundedness of the weighted Hardy-Cesàro operator $U_{\psi,s}$ and its commutator on generalized weighted Morrey spaces $M_{p,\varphi}(\omega)$. These results extend the results in (Xiao, 2001), (Fu, Liu, & Lu, 2009) and (Fu, & Lu, 2010) in some sense.

Throughout the paper, the letter C is used to denote (possibly different) constants that are independent of the essential variables. We denote a ball centered at x of radius r and its Lebesgue measure by $B(x,r)$ and $|B(x,r)|$, respectively. In addition, for each ball $B(x,r)$ and $t>0$, $tB(x,r)$ means $B(tx,tr)$.

2. Main results

In this section, we will first show the boundedness of the generalized Hardy-Cesàro operator $U_{\psi,s}$ on spaces $M_{p,\varphi}(\omega)$ for the class of weights ω and φ below.

Definition 2.1.

Let α be a real number. Then we denote by $\mathcal{W}_\alpha$ the set of all weight functions ω on $\mathbb{R}^n$ which are absolutely homogeneous of degree α, that is $\omega(tx) = |t|^\alpha \omega(x)$, for all $t \in \mathbb{R} \setminus \{0\}, x \in \mathbb{R}^n$ and $0 < \int\limits_{S_n} \omega(y) d\sigma(y) < \infty$, where $S_n = \{x \in \mathbb{R}^n : |x| = 1\}$.

Let us describe some typical examples and properties of $\mathcal{W}_\alpha$.

For a weight $\omega \in \mathcal{W}_\alpha$, by standard calculations, it is easy to see that $\omega \in L^1_{\text{loc}}(\mathbb{R}^n)$ if and only if $\alpha > -n$.

For $n \geq 1$ and $\alpha > -n$, $\omega(x) = |x|^\alpha$ is in $\mathcal{W}_\alpha$ and has *the doubling property*, that is *there exists a positive constant C such that $\omega(B(x,2r)) \leq C\omega(B(x,r))$, for all balls $B(x,r)$.*

In addition, if ω_1, ω_2 are in $\mathcal{W}_\alpha$, then so are $\theta\omega_1 + \lambda\omega_2$ for all $\theta, \lambda > 0$.

Definition 2.2.

Let $\beta > 0$ and φ be a positive measurable function on $\mathbb{R}^n \times (0,\infty)$. We say φ is a subhomogeneous function on $\mathbb{R}^n \times (0,\infty)$, denoted by $\varphi \in SH^\beta(\mathbb{R}^n \times (0,\infty))$, if there exists a positive constant C such that for all $(x,r) \in \mathbb{R}^n \times (0,\infty)$ and for all $t \in (0,\infty)$, one has $\varphi(tx,tr) \leq Ct^\beta \varphi(x,r)$.

We say φ is a weak subhomogeneous function on $\mathbb{R}^n \times (0,\infty)$, denoted by $\varphi \in WSH^\beta(\mathbb{R}^n \times (0,\infty))$, if there exists a positive constant C such that for all $r \in (0,\infty)$ and for all $t \in (0,\infty)$, one has $\varphi(0,tr) \leq Ct^\beta \varphi(0,r)$.

Examples of such functions are $\varphi(x,r) = r^\beta$ or homogeneous functions of degree β.

Our first main result in this section is formulated as follows.

Theorem 2.3.

Let $1 < p < \infty$, $s, \psi : [0,1] \to [0,\infty)$ be measurable functions such that $s(t) > 0$ a.e. $t \in [0,1]$, $\omega \in \mathcal{W}_\alpha$ for some $\alpha > -n$, and $\varphi \in SH^\beta(\mathbb{R}^n \times (0,\infty))$ for some $\beta > 0$. Then $U_{\psi,s}$ is bounded on $M_{p,\varphi}(\omega)$, provided that $\int\limits_0^1 s(t)^\beta \psi(t) dt < \infty$.

Proof.

Assume that $\int\limits_0^1 s(t)^\beta \psi(t) dt < \infty$.

For any $f \in M_{p,\varphi}(\omega)$, $x \in \mathbb{R}^n$ and $r > 0$, it follows from the Minkowski inequality that

$$[\varphi(x,r)]^{-1}\left([\omega(B(x,r))]^{-1}\int\limits_{B(x,r)}\left|U_{\psi,s}f(y)\right|^p\omega(y)dy\right)^{\frac{1}{p}}$$

$$=[\varphi(x,r)]^{-1}[\omega(B(x,r))]^{-\frac{1}{p}}\left(\int\limits_{B(x,r)}\left|\int_0^1 f\left(s(t)y\right)\psi(t)dt\right|^p\omega(y)dy\right)^{\frac{1}{p}}$$

$$\leq[\varphi(x,r)]^{-1}[\omega(B(x,r))]^{-\frac{1}{p}}\int_0^1\left(\int\limits_{B(x,r)}\left|f\left(s(t)y\right)\right|^p\omega(y)dy\right)^{\frac{1}{p}}\psi(t)dt$$

$$=[\varphi(x,r)]^{-1}[\omega(B(x,r))]^{-\frac{1}{p}}\int_0^1\left(\int\limits_{B(s(t)x,s(t)r)}\left|f\left(y\right)\right|^p\omega(y)dy\right)^{\frac{1}{p}}s(t)^{-\frac{n+\alpha}{p}}\psi(t)dt$$

$$\leq\int_0^1\frac{\varphi(s(t)x,s(t)r)}{\varphi(x,r)}[\varphi(s(t)x,s(t)r)]^{-1}[\omega(B(x,r))]^{-\frac{1}{p}}\left(\int\limits_{B(s(t)x,s(t)r)}\left|f\left(y\right)\right|^p\omega(y)dy\right)^{\frac{1}{p}}s(t)^{-\frac{n+\alpha}{p}}\psi(t)dt$$

$$\leq\|f\|_{M_{p,\varphi}(\omega)}\int_0^1\frac{\varphi(s(t)x,s(t)r)}{\varphi(x,r)}\psi(t)dt\leq C\|f\|_{M_{p,\varphi}(\omega)}\int_0^1 s(t)^{\beta}\psi(t)dt,$$

where the last inequality comes from the assumption that $\varphi \in SH^{\beta}(\mathbb{R}^n \times (0,\infty))$.

Clearly, the estimates above together imply

$$\left\|U_{\psi,s}f\right\|_{M_{p,\varphi}(\omega)}\leq C\|f\|_{M_{p,\varphi}(\omega)}\int_0^1 s(t)^{\beta}\psi(t)dt<\infty.$$

In other words, $U_{\psi,s}$ is defined as a bounded operator on $M_{p,\varphi}(\omega)$ and

$$\left\|U_{\psi,s}\right\|_{M_{p,\varphi}(\omega)\to M_{p,\varphi}(\omega)}\leq C\int_0^1 s(t)^{\beta}\psi(t)dt,$$

which completes the proof of Theorem 2.3.

Analogous to the proof of Theorem 2.3, we can give a sufficient condition such that the integral operator $\mathcal{V}_{\psi,s}$, which is defined by

$$\mathcal{V}_{\psi,s}f(x)=\int_0^{\infty}f\left(s(t)x\right)\psi(t)dt,$$

is bounded on $M_{p,\varphi}(\omega)$ as follows.

Theorem 2.4.

Let $1 < p < \infty$, *let* $s, \psi : [0,\infty) \to [0,\infty)$ *be measurable functions such that* $s(t) > 0$ *a.e.* $t \in [0,\infty)$, *let* $\omega \in \mathcal{W}_\alpha$ *for some* $\alpha > -n$, *and let* $\varphi \in SH^\beta(\mathbb{R}^n \times (0,\infty))$ *for some* $\beta > 0$. *Then* $\mathcal{V}_{\psi,s}$ *is bounded on* $M_{p,\varphi}(\omega)$, *provided that* $\int_0^\infty s(t)^\beta \psi(t)dt < \infty$.

The rest of this section is devoted to establishing the boundedness of generalized weighted Hardy-Cesàro commutators, with symbols in weighted central bounded mean oscillation spaces, on generalized weighted central Morrey spaces.

Let us recall here the definitions of weighted bounded mean oscillation spaces $BMO(\omega)$ and weighted central bounded mean oscillation spaces $CMO^p(\omega)$.

Definition 2.5.

The weighted bounded mean oscillation space $BMO(\omega)$ *is defined by*

$$BMO(\omega) := \left\{ f \in L^p_{\text{loc}}(\omega) : \| f \|_{BMO(\omega)} < \infty \right\},$$

where

$$\| f \|_{BMO(\omega)} = \sup_B \left(\frac{1}{\omega(B)} \int_B | f(x) - f_{B,\omega} | \omega(x)dx \right),$$

$$\omega(B) = \int_B \omega(x)dx$$

and $f_{B,\omega}$ *is the mean value of* f *on* B *with weight* ω, *namely*

$$f_{B,\omega} = \frac{1}{\omega(B)} \int_B f(x)\omega(x)dx.$$

Definition 2.6.

The weighted central bounded mean oscillation space $CMO^p(\omega)$, *for* $p \geq 1$, *is defined by*

$$CMO^p(\omega) := \left\{ f \in L^p_{\text{loc}}(\omega) : \| f \|_{CMO^p(\omega)} < \infty \right\},$$

where

$$\| f \|_{CMO^p(\omega)} = \sup_{r>0} \left(\frac{1}{\omega(B(0,r))} \int_{B(0,r)} | f(x) - f_{B(0,r),\omega} |^p \omega(x)dx \right)^{1/p}.$$

In the sequent, we will need the following key lemmas relating to $BMO(\omega)$ and $CMO^p(\omega)$ spaces.

Lemma 2.7.

Assume that ω is a weight function with the doubling property. Then for any $1 < p < \infty$, there exists some positive constant C_p such that

$$\| f \|_{BMO^p(\omega)} := \sup_B \left(\frac{1}{\omega(B)} \int_B \left| f(x) - f_{B,\omega} \right|^p \omega(x) dx \right)^{1/p} \le C_p \| f \|_{BMO(\omega)}.$$

Proof.

The proof of Lemma 2.7 is similar to the proof of Corollary 6.12 in (Duoandikoetxea, 2000) with slight modifications. So we omit the details here.

The next lemma describes the inclusions between spaces $CMO^p(\omega), p \ge 1$, and between $CMO^p(\omega)$ with $BMO(\omega)$.

Lemma 2.8.

(a) Let ω be a weight function. If $1 \le p < q < \infty$ then $CMO^q(\omega) \subset CMO^p(\omega)$ and for any $b \in CMO^q(\omega)$, we have $\| b \|_{CMO^p(\omega)} \le \| b \|_{CMO^q(\omega)}$.

(b) Assume in addition that ω holds the doubling property. Then $BMO(\omega) \subset CMO^p(\omega)$, for all $p \in [1, \infty)$. Moreover, for any $b \in BMO(\omega)$, there exists a positive constant C_p such that $\| b \|_{CMO^p(\omega)} \le C_p \| b \|_{BMO(\omega)}$.

Proof.

The part (a) of the lemma follows from the definitions of the spaces $CMO^p(\omega)$ and from the Holder inequality with the pair $\left(\frac{q}{p}, \left(\frac{q}{p} \right)' \right)$.

Let us now prove part (b). Indeed, in view of Lemma 2.7, if $b \in BMO(\omega)$ then there exists a positive constant C_p such that $\| b \|_{BMO^p(\omega)} \le C_p \| b \|_{BMO(\omega)}$.

On the other hand, it is clear to see that

$$\| b \|_{BMO^p(\omega)} \ge \sup_{r>0} \left(\frac{1}{\omega(B(0,r))} \int_{B(0,r)} |b(x) - b_{B(0,r),\omega}|^p \omega(x) dx \right)^{1/p} = \| b \|_{CMO^p(\omega)}.$$

The last two estimates then prove part (b) of this lemma.

Lemma 2.9.

Let ω be a doubling weight function. Then, there exists some positive constant C such that for any balls $B_1 = B(x_1, r_1)$, $B_2 = B(x_2, r_2)$, whose intersection is not empty, and

$\frac{1}{2}r_2 \le r_1 \le 2r_2$, *then* $\omega(B) \le C\omega(B_i)$, $i=1,2$. *Here,* B *is the smallest ball which contains both* B_1 *and* B_2. *Moreover, for each function* $b \in BMO(\omega)$, *we have*

$$\left|b_{B_1,\omega} - b_{B_2,\omega}\right| \le 2C \| b \|_{BMO(\omega)}.$$

Proof.

Since ω has the doubling property, there exists a constant C_1 such that $\omega\big(B(x,2r)\big) \le C_1\omega\big(B(x,r)\big)$, for any $x \in \mathbb{R}^n$ and $r > 0$. Without loss of generality, assume that $r_2 \le r_1 \le 2r_2$. Let $B_1 = B(x_1,r_1)$, $B_2 = B(x_2,r_2)$ be two balls whose intersection is not empty and $r_2 \le r_1 \le 2r_2$, and B be the smallest ball which contains both B_1 and B_2. Take $x \in B_1 \cap B_2$. Then,

$$\omega(B) \le \omega\big(B(x,2r_1)\big) \le C_1\omega\big(B(x,r_1)\big) \le C_1\omega\big(B(x_1,2r_1)\big) \le C_1^2\omega\big(B_1\big),$$

and

$$\omega(B) \le \omega\big(B(x,4r_2)\big) \le C_1^2\omega\big(B(x,r_2)\big) \le C_1^3\omega\big(B(x_2,r_2)\big).$$

We now choose the constant $C = \max\{C_1^2, C_1^3\}$.

On the other hand, one has

$$\left|b_{B_1,\omega} - b_{B_2,\omega}\right| \le \left|b_{B_1,\omega} - b_{B,\omega}\right| + \left|b_{B,\omega} - b_{B_2,\omega}\right|.$$

In the light of choosing the constant C, we deduce that

$$\left|b_{B,\omega} - b_{B_1,\omega}\right| = \left|b_{B,\omega} - \frac{1}{\omega(B_1)}\int_{B_1} b(y)\omega(y)dy\right|$$

$$\le \frac{1}{\omega(B_1)}\int_{B_1}\left|b(y) - b_{B,\omega}\right|\omega(y)dy \le \frac{C}{\omega(B)}\int_{B}\left|b(y) - b_{B,\omega}\right|\omega(y)dy \le C\| b \|_{BMO(\omega)}.$$

Finally, one estimates the left term in a similar way and completes the proof of Lemma 2.9.

Lemma 2.10.

Let ω *be a doubling weight function. Then, there exists some positive constant* C *such that for any balls* $B_1 = B(0,r_1)$, $B_2 = B(0,r_2)$, *and* $\frac{1}{2}r_2 \le r_1 \le 2r_2$, *then* $\omega(B) \le C\omega(B_i)$, $i = 1, 2$. *Here,* $B = B(0,r)$ *is the smallest ball which contains both* B_1 *and* B_2. *Moreover, for any function* $b \in CMO^p(\omega), p \ge 1$, *we have*

$$\left|b_{B_1,\omega} - b_{B_2,\omega}\right| \le 2C \| b \|_{CMO^p(\omega)}.$$

Proof.

Thanks to Lemma 2.9, it suffices to prove

$$\left|b_{B_1,\omega}-b_{B_2,\omega}\right|\le 2C\,\|\,b\,\|_{CMO^p(\omega)}.$$

Obviously, we have

$$\left|b_{B_1,\omega}-b_{B_2,\omega}\right|\le\left|b_{B_1,\omega}-b_{B,\omega}\right|+\left|b_{B,\omega}-b_{B_2,\omega}\right|.$$

One now can observe that

$$\left|b_{B,\omega}-b_{B_1,\omega}\right|=\left|b_{B,\omega}-\frac{1}{\omega(B_1)}\int_{B_1}b(y)\omega(y)dy\right|$$

$$\le\frac{1}{\omega(B_1)}\int_{B_1}\left|b(y)-b_{B,\omega}\right|\omega(y)dy\le\frac{C}{\omega(B)}\int_{B}\left|b(y)-b_{B,\omega}\right|\omega(y)dy\le C\,\|\,b\,\|_{CMO^p(\omega)},$$

where the last estimate follows from the Holder inequality for the pair $(p,\ p')$ if $p>1$.

One then can estimate the remaining term analogously to end the proof of Lemma 2.10.

We are now in a position to state the following main result.

Theorem 2.11.

Let $1<p<q<\infty$, $s,\psi:[0,1]\to[0,\infty)$ *be measurable functions such that* $0<s(t)\le 1$ *a.e.* $t\in[0,1]$, $\omega\in\mathcal{W}_\alpha$ *hold the doubling property for some* $\alpha>-n$, $\varphi\in WSH^{\beta}(\mathbb{R}^n\times(0,\infty))$ *for some* $\beta>0$, *and* $b\in CMO^{\lambda}(\omega)$, $\lambda\ge\lambda^*=\dfrac{qp}{q-p}$.

Then $U^b_{\psi,s}$ *is bounded from* $M^{cen}_{q,\varphi}(\omega)$ *to* $M^{cen}_{p,\varphi}(\omega)$, *provided that*

$$\int_0^1 s(t)^{\beta-\frac{n+\alpha}{q}}\left(2-\log_2 s(t)\right)\psi(t)dt<\infty.$$

Proof.

Suppose that $\displaystyle\int_0^1 s(t)^{\beta-\frac{n+\alpha}{q}}\left(2-\log_2 s(t)\right)\psi(t)dt<\infty.$

Let B be any ball centered at the origin of radius r, and let f be any function in $M^{cen}_{q,\varphi}(\omega)$. By applying the Minkowski inequality, we obtain

$$I=\left(\varphi(0,r)^{-p}\frac{1}{\omega(B)}\int_B\left|U^b_{\psi,s}f(y)\right|^p\omega(y)dy\right)^{\frac{1}{p}}$$

$$\le\int_0^1\left(\varphi(0,r)^{-p}\frac{1}{\omega(B)}\int_B\left|(b(y)-b(s(t)y))f(s(t)y)\right|^p\omega(y)dy\right)^{\frac{1}{p}}\psi(t)dt.$$

At this point, in use of the Minkowski's triangle inequality to the right-hand side of the above estimate, it is clear to see that

$$I \le C(I_1 + I_2 + I_3),$$

where

$$I_1 = \int_0^1 \left(\varphi(0,r)^{-p} \frac{1}{\omega(B)} \int_B \left| (b(y) - b_{B,\omega}) f(s(t)y) \right|^p \omega(y)dy \right)^{\frac{1}{p}} \psi(t)dt,$$

$$I_2 = \int_0^1 \left(\varphi(0,r)^{-p} \frac{1}{\omega(B)} \int_B \left| (b_{s(t)B,\omega} - b_{B,\omega}) f(s(t)y) \right|^p \omega(y)dy \right)^{\frac{1}{p}} \psi(t)dt,$$

$$I_3 = \int_0^1 \left(\varphi(0,r)^{-p} \frac{1}{\omega(B)} \int_B \left| (b(s(t)y) - b_{s(t)B,\omega}) f(s(t)y) \right|^p \omega(y)dy \right)^{\frac{1}{p}} \psi(t)dt,$$

and the constant C depends only on p.

Let us now estimate the term I_1. It follows from the Holder inequality with the pair $\left(l = \frac{q}{p}, l' = \frac{q}{q-p} \right)$ for the term I_1 that

$$I_1 \le \varphi(0,r)^{-1} \int_0^1 \left(\frac{1}{\omega(B)} \int_B |f(s(t)y)|^q \omega(y)dy \right)^{\frac{1}{q}} \left(\frac{1}{\omega(B)} \int_B |b(y) - b_{B,\omega}|^{\lambda^*} \omega(y)dy \right)^{\frac{1}{\lambda^*}} \psi(t)dt.$$

Due to Lemma 2.8, we then deduce that

$$I_1 \le \|b\|_{CMO^{\lambda}(\omega)} \varphi(0,r)^{-1} \int_0^1 \left(\frac{1}{\omega(B)} \int_B |f(s(t)y)|^q \omega(y)dy \right)^{\frac{1}{q}} \psi(t)dt$$

$$\le C \|b\|_{CMO^{\lambda}(\omega)} \|f\|_{M^{cen}_{q,\varphi}(\omega)} \int_0^1 s(t)^{\beta - \frac{n+\alpha}{q}} \psi(t)dt$$

$$\le C \|b\|_{CMO^{\lambda}(\omega)} \|f\|_{M^{cen}_{q,\varphi}(\omega)} \int_0^1 s(t)^{\beta - \frac{n+\alpha}{q}} \left(2 - \log_2 s(t)\right) \psi(t)dt.$$

Similarly, one can use the same argument above to have

$$I_3 \le C \|b\|_{CMO^{\lambda}(\omega)} \|f\|_{M^{cen}_{q,\varphi}(\omega)} \int_0^1 s(t)^{\beta - \frac{n+\alpha}{q}} \left(2 - \log_2 s(t)\right) \psi(t)dt.$$

For the term I_2, rewrite this term as

$$I_2 = \int_0^1 \left(\varphi(0,r)^{-p} \frac{1}{\omega(B)} \int_B |f(s(t)y)|^p \, \omega(y)dy \right)^{\frac{1}{p}} \left| b_{B,\omega} - b_{s(t)B,\omega} \right| \psi(t)dt.$$

Then we employ the Holder inequality with the pair $\left(l = \frac{q}{p}, l' = \frac{q}{q-p} \right)$ for this term to get

$$I_2 \le \int_0^1 \left(\varphi(0,r)^{-p} \frac{1}{\omega(B)} \int_B |f(s(t)y)|^q \, \omega(y)dy \right)^{\frac{1}{q}} \left| b_{B,\omega} - b_{s(t)B,\omega} \right| \psi(t)dt$$

$$\le C \| f \|_{M_{q,\varphi}^{cen}(\omega)} \int_0^1 \left| b_{B,\omega} - b_{s(t)B,\omega} \right| s(t)^{\beta - \frac{n+\alpha}{q}} \psi(t)dt$$

$$\le C \| f \|_{M_{q,\varphi}^{cen}(\omega)} \sum_{m=0}^{\infty} \int_{\left\{ t \in [0,1]: 2^{-m-1} \le s(t) \le 2^{-m} \right\}} \left| b_{B,\omega} - b_{s(t)B,\omega} \right| s(t)^{\beta - \frac{n+\alpha}{q}} \psi(t)dt.$$

At this stage, observe that for each $m \in \mathbb{N}$, we have

$$\left| b_{B,\omega} - b_{s(t)B,\omega} \right| \le \sum_{i=0}^{m} \left| b_{2^{-i-1}B,\omega} - b_{2^{-i}B,\omega} \right| + \left| b_{2^{-m-1}B,\omega} - b_{s(t)B,\omega} \right|.$$

Therefore, in light of Lemma 2.10, we deduce that

$$I_2 \le C \|b\|_{CMO^{\lambda}(\omega)} \| f \|_{M_{q,\varphi}^{cen}(\omega)} \sum_{m=0}^{\infty} \int_{\left\{ t \in [0,1]: 2^{-m-1} \le s(t) \le 2^{-m} \right\}} (m+2) s(t)^{\beta - \frac{n+\alpha}{q}} \psi(t)dt$$

$$\le C \|b\|_{CMO^{\lambda}(\omega)} \| f \|_{M_{q,\varphi}^{cen}(\omega)} \sum_{m=0}^{\infty} \int_{\left\{ t \in [0,1]: 2^{-m-1} \le s(t) \le 2^{-m} \right\}} \left(2 - \log_2 s(t) \right) s(t)^{\beta - \frac{n+\alpha}{q}} \psi(t)dt$$

$$\le C \|b\|_{CMO^{\lambda}(\omega)} \| f \|_{M_{q,\varphi}^{cen}(\omega)} \int_0^1 s(t)^{\beta - \frac{n+\alpha}{q}} \left(2 - \log_2 s(t) \right) \psi(t)dt,$$

which, combined with the last estimates of I_1 and I_3 above, completes the proof of Theorem 2.11.

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CÁC BẤT ĐẲNG THỨC CÓ TRỌNG VỀ CHUẨN CỦA TOÁN TỬ VÀ HOÁN TỬ HARDY-CESÀRO TỔNG QUÁT TRÊN CÁC KHÔNG GIAN MORREY CÓ TRỌNG TỔNG QUÁT VỚI CÁC BIỂU TƯỢNG TRONG KHÔNG GIAN CMO

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TÓM TẮT

Trong bài báo này, mục đích chính của chúng tôi là nghiên cứu tính bị chặn của hoán tử Hardy-Cesàro có trọng trên các không gian Morrey tổng quát $M_{p,\varphi}(\omega)$. Chúng tôi thiết lập được một số điều kiện đủ cho tính bị chặn của toán tử Hardy-Cesàro và hoán tử của nó trên các không gian Morrey có trọng tổng quát $M_{p,\varphi}(\omega)$ khi các biểu tượng thuộc không gian CMO.

Từ khóa: toán tử Hardy-Cesàro có trọng; hoán tử; không gian Morrey có trọng tổng quát; không gian CMO