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Research Article

CALCULATION OF SCALAR SCATTERING ON A PROLATE SPHEROID

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ABSTRACT

Method of separation of variables plays an important role in mathematical physics problems, especially in the scattering problem containing hyperbolic equations with the survey domain limited by coordinate surfaces of arbitrary shapes. In this paper, the method of separation of variables in the spherical coordinate system is developed for calculating the scalar stationary scattering problem on a prolate spheroid with an arbitrary ratio between wavelength and size of the spheroid.

Keywords: prolate spheroid; scattering; separation of variables

1. Introduction

Generally, the problem of scattering is extremely difficult and there have been few problems whose solutions can be expressed in analytical form. There have been a number of studies on spheroids (oblate and prolate) by (King, & Van Buren, 1972; Bowman et al., 1969; Handelman, & Sidman, 1972) and many others. However, the solution has always been in terms of spheroidal wave functions (radial and angular functions).

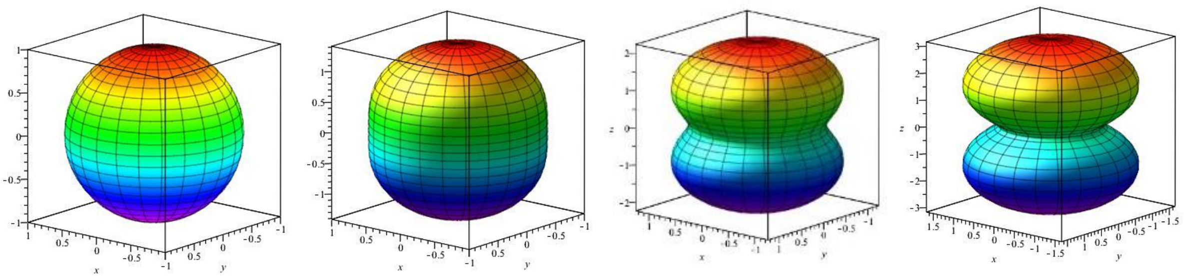

Figure 1. *Shapes of a prolate spheroid with* $\varepsilon^2 = 0, 1, 2, 3$ *(from left to right)*

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In this work we consider the axial symmetry scalar stationary scattering problem on a prolate spheroid by using the method of separation of variables. Prolate spheroid is an oblong ellipsoid whose semi-major axis is *Oz* (Acho, 1992). Choosing semi-minor axes of prolate spheroid for a unit length, the equation of a prolate spheroid in a spherical coordinate system has the form (Meixner, 1959; Flammer, 1962):

$$r = r(\theta) \equiv \sqrt{1+\varepsilon^2 \cos^2\theta},$$

where ε is a parameter depending on the ratio between the semi-major and semi-minor axis of a prolate spheroid. Shapes of a prolate spheroid are shown on Figure 1 at $\varepsilon^2 = 0, 1, 2, 3$.

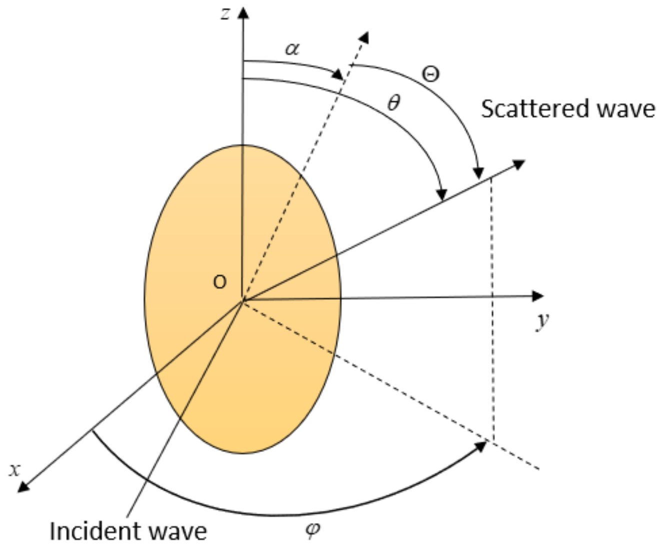

Figure 2. *Scattering geometry for a prolate spheroid.*

On Figure 2 the scattered field in the far-field zone is represented in the spherical coordinate system (r,θ,φ). Θ is the scattering angle. The origin of the Cartesian coordinate system is at the center of the spheroid while the *Oz* axis coincides with its axis of revolution. α is the incident angle (the angle between the direction of the incident wave and the *Oz* axis in the *Oxy* plane).

2. Method of separation of variables

In the Helmholts equation $\Delta\psi + k^2\psi = 0$, where $r > r_{\max} = \sqrt{1+\varepsilon^2}$ (outside the spheroid with $\theta = 0$ or $\theta = \pi$, then $\cos^2\theta = 1$) the wave function ψ is expanded into series:

$$\psi = \sum_{n=0}^{\infty}\left[c_n^+\chi_n^+(r) - c_n^-\chi_n^-(r)\right] Y_n(\theta),$$

where $Y_n(\theta) = \frac{\sqrt{2n+1}}{2} P_n(\cos\theta)$, $\chi_n^{\pm}(r) = \sqrt{\frac{\pi}{2kr}} H_{n+\frac{1}{2}}^{(1)(2)}(kr)$, P_n – Legendre polynomial, $H_{n+\frac{1}{2}}^{(1)(2)}(kr)$ – Hankel function of the first and the second kind.

Due to $\chi_n^{\pm}(r) = \chi_n^{\pm *}(r)$, $\chi_n^+\chi_n^{'-} - \chi_n^{+'}\chi_n^- = \frac{2}{ikr^2}$, thus $\chi_n^- Y_n$ and $\chi_n^+ Y_n$ are incident and outgoing waves, respectively.

In scattering problem, the amplitudes of incident wave c_n^- are given, and the ones of outgoing wave c_n^+ are expressed by c_n^- with the boudary condition on the surface of the scattering spheroid, i.e: $\psi(\theta, r(\theta)) = 0$, then: $-ik\pi\sqrt{1+\varepsilon^2\sin^2\theta}.\frac{\partial\psi}{\partial n}\Big|_{r=r(\theta)} \equiv q(\theta)$.

For the coefficients $c_n^{\pm}$ we have: $c_n^{\pm} = \int_0^{\pi} q(\theta)\,\chi_n^{\mp}[r(\theta)]Y_n(\theta)\sin\theta\, d\theta.$ (1)

Based on the expression (1), we need to find c_n^+ with the given value of c_n^- and the unknown quantity $q(\theta)$.

First, we expand the function $q(\theta)$ into a series of spherical functions:

$$q(\theta) = \sum_{m=0}^{\infty} \alpha_m Y_m(\theta). \quad (2)$$

Substitute (2) in (1) and denote: $a_{mn}^{\pm} = \int_0^{\pi} \chi_n^{\mp}[r(\theta)]\, Y_n(\theta) Y_m(\theta)\sin\theta\, d\theta,$ we obtain an infinite system of algebraic equations for the coefficients of α_m:

$$\sum_{m=0}^{\infty} a_{mn}^- \alpha_m = c_n^-, \quad n = 0, 1, 2, ... \quad (3)$$

After solving this system we obtain α_m, then the amplitudes c_n^+ are calculated by the formula:

$$c_n^+ = \sum_{m=0}^{\infty} a_{mn}^+ \alpha_m.$$

With the obtained values of c_n^+, the differential scattering cross-section can be calculated by the following formula:

$$I(\theta) = \lim_{r\to\infty} \frac{r^2}{2ik}\left(\psi_+^* \frac{\partial \psi_+}{\partial r} - \psi_+ \frac{\partial \psi_+^*}{\partial r}\right), \text{ here } \psi_+ = \sum_{n=0}^{\infty} c_n^+ \chi_n^+(r) Y_n(\theta).$$

Due to $\chi_n^+ \approx \frac{1}{kr} i^{-n-1} e^{ikr}$ at the large of r we have:

$$I(\theta) = | f(\theta)|^2, \quad f(\theta) = \frac{1}{ik} \sum_{n=0}^{\infty} i^{-n} c_n^+ Y_n(\theta). \tag{4}$$

The main difficulty of solving this problem is the infinite number of equations of the system (3). However, in principle the system (3) can be considered finite because effective scattering can be calculated in the case of a finite number of waves, i.e: $n \le N \approx k\sqrt{1+\varepsilon^2} + 3.$

This is easily confirmed in the case $\varepsilon = 0$, then $a_{mn}^{\pm} = \chi_n^{\mp}(1)\delta_{mn}$, $c_n^+ = \frac{c_n^- \chi_n^-(1)}{\chi_n^+(1)}$.

At the large of n $(n > k)$ we have: $\frac{c_n^+}{c_n^-} \approx 1 - i\exp[-(2n+1)(\alpha - ch\alpha)]$, where $ch\alpha = \frac{n+\frac{1}{2}}{k}$.

Table 1. *The values of* c_n^- *and* $|c_n^-|^2$ *at* $n = 0, 1, 2, 3, 4, 5$

n	0	1	2	3	4	5
c_n^-	-0.7071	-1.2247i	1.5811	1.8708i	-2.1213	-2.3452i
$\|c_n^-\|^2$	0.5	1.5	2.5	3.5	4.5	5.5

3. Calculation of scattering

In this section with given values of the amplitudes of the incident wave c_n^- we will calculate the amplitudes of outgoing wave c_n^+ and differential scattering cross-section.

Suppose that the incident wave is plane and has the form $\psi_0 = \exp(ikr\cos\theta)$ and c_n^- can be given in the form: $c_n^- = -i^n\sqrt{\frac{2n+1}{2}}$.

The values of c_n^- with corresponding squared modulus $|c_n^-|^2$ are evaluated and shown in Table 1.

Table 2. *The values of* c_n^+, $\left|c_n^+\right|^2$, S_N^+ $(n=\overline{0,N},\ N=4)$ *and* σ *at* $k=\frac{1}{2}$ *with* $\varepsilon^2=0, 1, 2, 3$

	$\varepsilon^2=0$	$\varepsilon^2=1$	$\varepsilon^2=2$	$\varepsilon^2=3$
c_0^+	0.382-0.595*i*	0.268-0.633*i*	0.193-0.648*i*	0.111-0.644*i*
c_1^+	0.089+1.222*i*	0.173+1.212*i*	0.268+1.195*i*	0.365+1.168*i*
c_2^+	-1.581+0.002*i*	-1.587-0.002*i*	-1.595-0.004*i*	-1.604-0.002*i*
c_3^+	-1.871*i*	0.001-1.871*i*	0.003-1.871*i*	0.006-1.182*i*
c_4^+	1.121	2.121	1.121	2.121
$\left\|c_0^+\right\|^2$	0.5	0.483	0.458	0.427
$\left\|c_1^+\right\|^2$	1.5	1.5	1.499	1.498
$\left\|c_2^+\right\|^2$	2.5	2.517	2.543	2.573
$\left\|c_3^+\right\|^2$	3.5	3.5	3.501	3.503
$\left\|c_4^+\right\|^2$	4.5	4.5	4.5	4.5
s_4^+	12.5	12.5	12.5	12.501
σ	0.935	1.215	1.514	1.815

The values of c_n^+ with corresponding squared modulus $|c_n^+|^2$ are shown in Table 2 and Table 3.

The solution of the problem depends on two parameters: k and ε. We choose $k=\frac{1}{2}$ and $k=1$ with $\varepsilon^2=0,1,2,3$. (Senior, 1960; Barlow, & Einspruch, 1961).

Table 3. *The values of* c_n^+, $\left|c_n^+\right|^2$, S_N^+ $(n=\overline{0,N},\ N=5)$ *and* σ *at* $k=1$ *with* $\varepsilon^2=0, 1, 2, 3$

	$\varepsilon^2=0$	$\varepsilon^2=1$	$\varepsilon^2=2$	$\varepsilon^2=3$
c_0^+	-0.294-0.643*i*	0.350-0.456*i*	-0.315-0.284*i*	-0.220-0.165*i*
c_1^+	0.510+1.114*i*	0.806+0.908*i*	1.013+0.626*i*	1.111+0.316*i*
c_2^+	-1.582+0.054*i*	-1.629+0.119*i*	-1.66+0.249*i*	-1.654+0.427*i*

c_3^+	-0.002-1.871i	0.013-1.878i	0.032-1.898i	0.051-1.932i
c_4^+	2.121	2.121+0.002i	2.121+0.99i	2.123+0.021i
$\lvert c_0^+\rvert^2$	0.5	0.332	0.18	0.075
$\lvert c_1^+\rvert^2$	1.5	1.474	1.418	1.335
$\lvert c_2^+\rvert^2$	2.5	2.669	2.819	2.917
$\lvert c_3^+\rvert^2$	3.5	3.529	3.604	3.736
$\lvert c_4^+\rvert^2$	4.5	4.5	4.501	4.509
$\lvert c_5^+\rvert^2$	5.5	5.5	5.499	5.496
s_5^+	18	18	18	18.006
σ	0.846	1.044	1.271	1.511

The quantity $S_N^{\pm} = \sum_{n=0}^{N} |c_n^{\pm}|^2$ characterizes the consumption of the flow of incident and outgoing waves. The values of this quantity are evaluated and expressed in Tables 2 and 3.

For calculating the differential scattering cross-section, the loss part of the series (4) is expressed in the form: $\frac{1}{ik}\delta(1-\cos\theta)$.

The quantity $f(\theta)$ at $\theta \neq 0$ is considered as a sum of series:

$\frac{1}{k}\sum_{n=0}^{\infty}\left[c_n^+ - i^n\sqrt{\frac{2n+1}{2}}\right] i^{-n-1} Y_n(\theta)$, the coefficients of which gradually decrease at $n \approx N$.

The total scattering cross section σ is calculated by the following formula:

$$2\pi\sigma = \int_0^{2\pi}\int_0^{\pi} I(\theta)\sin\theta d\theta d\varphi .$$

The values of σ are expressed in table 2 and Table 3.

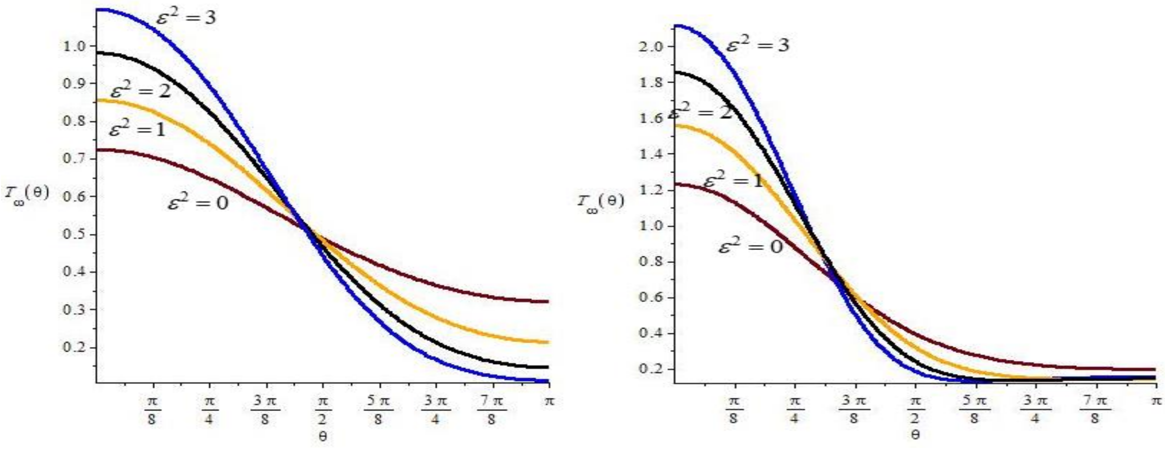

Figure 3. *Normalized scattering function $T_\omega(\theta)$ at $k=\frac{1}{2}$ (left) and $k=1$ (right)*

The normalized scattering function $T_\omega(\theta)$ is expressed in the form (Wallander, 1963):

$$T_\omega(\theta)=\frac{I(\theta)}{\sigma},$$

and is shown on Figure 3 at $k=\frac{1}{2}$ and $k=1$ with $\varepsilon^2=0, 1, 2, 3$.

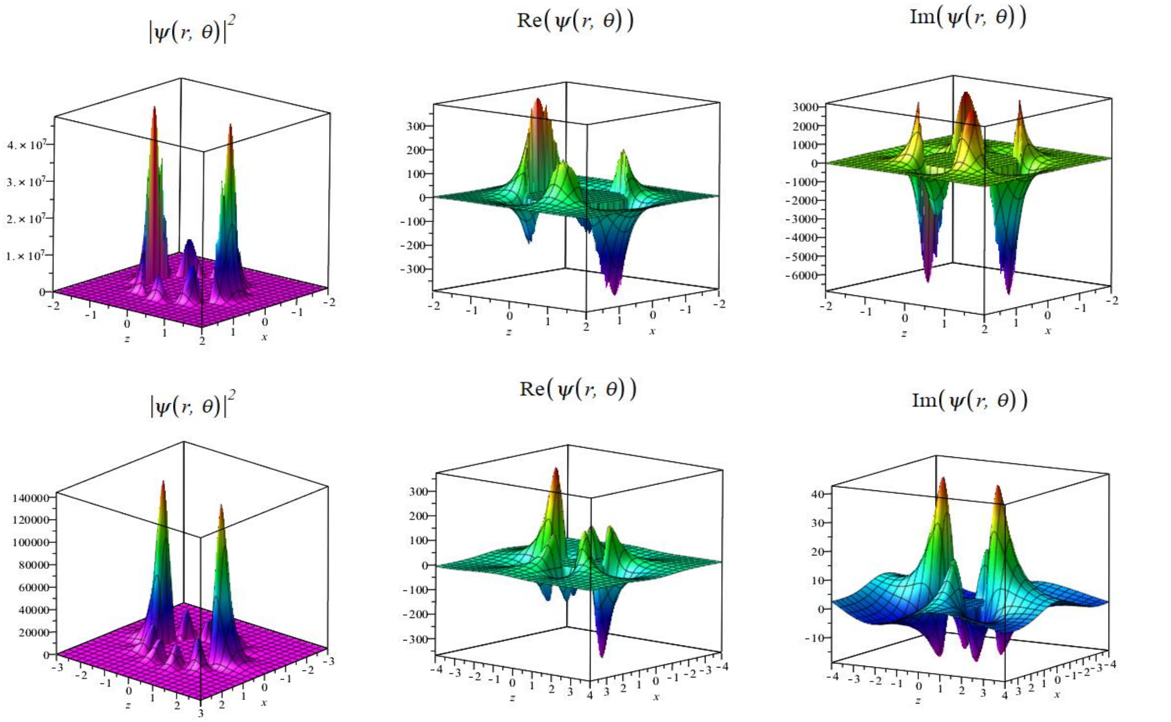

Figure. 4. *Probability density function $|\psi(r,\theta)|^2$ (left), real $\mathrm{Re}(\psi(r,\theta))$ (middle) and imaginary part $\mathrm{Im}(\psi(r,\theta))$ (right) of the scattering wave function $\psi(r,\theta)$ at $k=\frac{1}{2}$ with $\varepsilon^2=1$ (upper panel) and $k=1$ with $\varepsilon^2=3$ (lower panel)*

Probability density function $|\psi(r,\theta)|^2$, real $\mathrm{Re}(\psi(r,\theta))$ and imaginary part $\mathrm{Im}(\psi(r,\theta))$ of the scattering wave function $\psi(r,\theta)$ are shown on Figure 4.

4. Conclusion

The calculation results have shown that the method of separation of variables developed in this work gives practical convergence for solving scattering problems containing Helmholts equation. In general, this method can be applied for calculating boundary scattering and diffraction problems on objects of arbitrary shapes in the case of medium and long waves. Furthermore, the calculation results obtained in this problem can be compared to the ones calculated in an analytical form in a spheroidal coordinate system of other works.

❖ ***Conflict of Interest:*** *Authors have no conflict of interest to declare.*

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TÍNH TOÁN TÁN XẠ VÔ HƯỚNG TRÊN PHỎNG CẦU DÀI

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TÓM TẮT

Phương pháp tách biến đóng vai trò quan trọng trong các bài toán vật lí toán, đặc biệt trong bài toán tán xạ có chứa phương trình dạng hyperbolic khi miền khảo sát được giới hạn bởi các bề mặt tọa độ có hình dạng bất kì. Trong bài báo này, chúng ta sẽ sử dụng phương pháp tách biến trong hệ tọa độ cầu cho việc tính toán tán xạ dừng vô hướng trên một phỏng cầu dài với tỉ lệ bất kì giữa bước sóng và kích thước của phỏng cầu.

Từ khóa: phỏng cầu dài; tách biến; tán xạ