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Research Article

STRONG CONVERGENCE OF INERTIAL HYBRID ITERATION FOR TWO ASYMPTOTICALLY *G*-NONEXPANSIVE MAPPINGS IN HILBERT SPACE WITH GRAPHS

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ABSTRACT

In this paper, by combining the shrinking projection method with a modified inertial S-iteration process, we introduce a new inertial hybrid iteration for two asymptotically G-nonexpansive mappings and a new inertial hybrid iteration for two G-nonexpansive mappings in Hilbert spaces with graphs. We establish a sufficient condition for the closedness and convexity of the set of fixed points of asymptotically G-nonexpansive mappings in Hilbert spaces with graphs. We then prove a strong convergence theorem for finding a common fixed point of two asymptotically G-nonexpansive mappings in Hilbert spaces with graphs. By this theorem, we obtain a strong convergence result for two G-nonexpansive mappings in Hilbert spaces with graphs. These results are generalizations and extensions of some convergence results in the literature, where the convexity of the set of edges of a graph is replaced by coordinate-convexity. In addition, we provide a numerical example to illustrate the convergence of the proposed iteration processes.

Keywords: asymptotically *G*-nonexpansive mapping; Hilbert space with graphs; inertial hybrid iteration

1. Introduction and preliminaries

In 2012, by using the combination concepts between the fixed point theory and the graph theory, Aleomraninejad, Rezapour, and Shahzad (2012) introduced the notions of *G*-contractive mapping and *G*-nonexpansive mapping in a metric space with directed graphs and stated the convergence for these mappings. After that, there were many convergence results for *G*-nonexpansive mappings by some iteration processes established in Hilbert spaces and Banach spaces with graphs. In 2018, Sangago, Hunde, and Hailu (2018) introduced the notion of an asymptotically *G*-nonexpansive mapping and proved the weak and strong convergence of a modified Noor iteration process to common fixed points of a finite family of asymptotically *G*-nonexpansive mappings in Banach spaces with graphs. After that some authors proposed a two-step iteration process for two asymptotically *G*-nonexpansive mappings $T_1, T_2 : \Omega \to \Omega$ (Wattanataweekul, 2018) and a three-step iteration process for three asymptotically *G*-nonexpansive mappings $T_1, T_2, T_3 : \Omega \to \Omega$ (Wattanataweekul, 2019) as follows:

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$$u_1 \in \Omega, \begin{cases} v_n = (1-b_n)u_n + b_n T_2^n u_n \\ u_{n+1} = (1-a_n)v_n + a_n T_1^n v_n, \end{cases} \tag{1.1}$$

$$u_1 \in \Omega, \begin{cases} w_n = (1-c_n)u_n + c_n T_3^n u_n \\ v_n = (1-b_n)w_n + b_n T_2^n w_n \\ u_{n+1} = (1-a_n)v_n + a_n T_1^n v_n, \end{cases} \tag{1.2}$$

where $\{a_n\}, \{b_n\}, \{c_n\} \subset [0,1]$. Furthermore, the authors also established the weak and strong convergence results of the iteration process (1.1) and the iteration process (1.2) to common fixed points of asymptotically *G*-nonexpansive mappings in Banach spaces with graphs.

Currently, there were many methods to construct new iteration processes which generalize some previous iteration processes. In 2008, Mainge proposed the inertial Mann iteration by combining the Mann iteration and the inertial term $\gamma_n(u_n - u_{n-1})$. In 2018, by combining the CQ-algorithm and the inertial term, Dong, Yuan, Cho, and Rassias (2018) studied an inertial CQ-algorithm for a non-expansive mapping as follows:

$$u_1, u_2 \in H, \begin{cases} w_n = u_n + \gamma_n(u_n - u_{n-1}) \\ v_n = (1-a_n)w_n + a_n T w_n \\ C_n = \{v \in H : \| v_n - v \| \le \| w_n - v \|\} \\ Q_n = \{v \in H : \langle u_n - v, u_n - u_1 \rangle \le 0\} \\ u_{n+1} = P_{C_n \cap Q_n} u_1, \end{cases}$$

where $\{a_n\} \subset [0,1]$, $\{\gamma_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in \mathbb{R}$, $T : H \to H$ is a nonexpansive mapping, and $P_{C_n \cap Q_n} u_1$ is the metric projection of u_1 onto $C_n \cap Q_n$.

In 2019, by combining a modified S-iteration process with the inertial extrapolation, Phon-on, Makaje, Sama-Ae, and Khongraphan (2019) introduced an inertial *S*-iteration process for two nonexpansive mappings such as:

$$u_1, u_2 \in H, \begin{cases} w_n = u_n + \gamma_n(u_n - u_{n-1}) \\ v_n = (1-a_n)w_n + a_n T_1 w_n \\ u_{n+1} = (1-b_n)T_1 w_n + b_n T_2 v_n. \end{cases}$$

where $\{a_n\}, \{b_n\} \subset [0,1]$, $\{\gamma_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in \mathbb{R}$, and $T_1, T_2 : H \to H$ are two nonexpansive mappings. Recently, by combining the shrinking projection method with a modified *S*-iteration process, Hammad, Cholamjiak, Yambangwai, and Dutta (2019) introduced the following hybrid iteration for two *G*-nonexpansive mappings

$$u_1 \in \Omega, \Omega_1 = \Omega, \begin{cases} v_n = (1-b_n)u_n + b_n T_1 u_n \\ w_n = (1-a_n)T_1 v_n + a_n T_2 v_n \\ \Omega_{n+1} = \{w \in \Omega_n : \| w_n - w \| \le \| u_n - w \|\} \\ u_{n+1} = P_{\Omega_{n+1}} u_1, \end{cases} \tag{1.3}$$

where $\{a_n\}, \{b_n\} \subset [0,1]$, $T_1, T_2 : \Omega \to \Omega$ are two *G*-nonexpansive mappings, and $P_{\Omega_{n+1}} u_1$ is the metric projection of u_1 onto Ω_{n+1}.

Motivated by these works, we introduce an iteration process for two *G*-nonexpansive mappings $T_1, T_2 : H \to H$ such as:

$$u_1, u_2 \in H, \Omega_1 = H, \begin{cases} z_n = u_n + \gamma_n (u_n - u_{n-1}) \\ v_n = (1-b_n) z_n + b_n T_1 z_n \\ w_n = (1-a_n) T_1 v_n + a_n T_2 v_n \\ \Omega_{n+1} = \{ w \in \Omega_n : \| w_n - w \| \leq \| z_n - w \| \} \\ u_{n+1} = P_{\Omega_{n+1}} u_1, \end{cases} \tag{1.4}$$

and an iteration process for two asymptotically *G*-nonexpansive mappings $T_1, T_2 : H \to H$ such as:

$$u_1, u_2 \in H, \Omega_1 = H, \begin{cases} z_n = u_n + \gamma_n (u_n - u_{n-1}) \\ v_n = (1-b_n) z_n + b_n T_1^n z_n \\ w_n = (1-a_n) T_1^n v_n + a_n T_2^n v_n \\ \Omega_{n+1} = \{ w \in \Omega_n : \| w_n - w \|^2 \leq \| z_n - w \|^2 + \varepsilon_n \} \\ u_{n+1} = P_{\Omega_{n+1}} u_1 \end{cases} \tag{1.5}$$

where $\{a_n\}, \{b_n\} \subset [0,1]$, $\{\gamma_n\} \subset [\alpha, \beta]$ for some $\alpha, \beta \in \mathbb{R}$, H is a real Hilbert space, $P_{\Omega_{n+1}} u_1$ is the metric projection of u_1 onto Ω_{n+1}, and ε_n is defined in Theorem 2.2 in Section 2. Then, under some conditions, we prove that the sequence $\{u_n\}$ generated by (1.5) strongly converges to the projection of the initial point u_1 onto the set of all common fixed points of T_1 and T_2 in Hilbert spaces with graphs. By this theorem, we obtain a strong convergence result for two *G*-nonexpansive mappings by the iteration process (1.4) in Hilbert spaces with graphs. In addition, we give a numerical example for supporting obtained results.

We now recall some notions and lemmas as follows:

Throughout this paper, let $G = (V(G), E(G))$ be a directed graph, where the set all vertices and edges denoted by $V(G)$ and $E(G)$, respectively. We assume that all directed graphs are reflexive, that is, $(u,u) \in E(G)$ for each $u \in V(G)$, and G has no parallel edges. A directed graph $G = (V(G), E(G))$ is said to be *transitive* if for any $u, v, w \in V(G)$ such that (u,v) and (v,w) are in $E(G)$, then $(u,w) \in E(G)$.

Definition 1.1.

Tiammee, Kaewkhao, & Suantai (2015, p.4): Let X be a normed space, Ω be a nonempty subset of X, and $G = (V(G), E(G))$ be a directed graph such that $V(G) = \Omega$. Then Ω is said to have *property* (G) if for any sequence $\{u_n\}$ in Ω such that $(u_n, u_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$ and $\{u_n\}$ weakly converging to $u \in \Omega$, then there exists a subsequence $\{u_{n(k)}\}$ of $\{u_n\}$ such that $(u_{n(k)}, u) \in E(G)$ for all $k \in \mathbb{N}$.

Definition 1.2.

Nguyen, & Nguyen (2020): Definition 3.1: Let X be a normed space and $G = (V(G), E(G))$ be a directed graph such that $E(G) \subset X \times X$. The set of edges $E(G)$ is said to be *coordinate-convex* if for all $(p,u), (p,v), (u,p), (v,p) \in E(G)$ and for all $t \in [0,1]$, then $t(p,u) + (1-t)(p,v) \in E(G)$ and $t(u,p) + (1-t)(v,p) \in E(G)$.

Definition 1.3.

Tripak (2016) - Definition 2.1 and Sangago et al. (2018)- Definition 3.1: Let X be a normed space, $G=(V(G),E(G))$ be a directed graph such that $V(G)\subset \mathrm{X},$ and $T:V(G)\to V(G)$ be a mapping. Then

(1) T is said to be *G-nonexpansive* if

(a) T is *edge-preserving*, that is, for all $(u,v)\in E(G),$ we have $(Tu,Tv)\in E(G).$

(b) $\|Tu-Tv\|\le\|u-v\|,$ whenever $(u,v)\in E(G)$ for any $u,v\in V(G).$

(2) T is call *asymptotically* G*-nonexpansive* mapping if

(a) T is edge-preserving.

(b) There exists a sequence $\{\lambda_n\}\subset[1,\infty)$ with $\sum_{n=1}^{\infty}(\lambda_n-1)<\infty$ such that $\|T^n u-T^n v\|\le\lambda_n\|u-v\|$ for all $n\in\mathbb{N},$ whenever $(u,v)\in E(G)$ for any $u,v\in V(G),$ where $\{\lambda_n\}$ is said to be an *asymptotic coefficient sequence.*

Remark 1.4.

Every *G*-nonexpansive mapping is an asymptotically *G*-nonexpansive mapping with the asymptotic coefficients $\lambda_n=1$ for all $n\in\mathbb{N}.$

Lemma 1.5.

Sangago et al. (2018) - Theorem 3.3: *Let Ω be a nonempty closed, convex subset of a real Banach space $X,$ Ω have Property $(G),$ $G=(V(G),E(G))$ be a directed graph such that $V(G)=\Omega,$ $T:\Omega\to\Omega$ be an asymptotically G-nonexpansive mapping, $\{u_n\}$ be a sequence in Ω converging weakly to $u\in\Omega,$ $(u_n,u_{n+1})\in E(G)$ and $\lim_{n\to\infty}\|Tu_n-u_n\|=0.$ Then $Tu=u.$*

Let H be a real Hilbert space with inner product $\langle .,.\rangle$ and norm $\|.\|,\Omega$ be a nonempty, closed and convex subset of a Hilbert space $H.$ Now, we recall some basic notions of Hilbert spaces which we will use in the next section.

The nearest point projection of H onto Ω is denoted by $P_\Omega,$ that is, for all $u\in H,$ we have $\|u-P_\Omega u\|=\inf\{\|u-v\|:v\in\Omega\}.$ Then P_Ω is called *the metric projection* of H onto $\Omega.$ It is known that for each $u\in H,$ $p=P_\Omega u$ is equivalent to $\langle u-p,p-v\rangle\ge 0$ for all $v\in\Omega.$

Lemma 1.6.

Alber (1996, p.5): *Let H be a real Hilbert space, Ω be a nonempty, closed and convex subset of $H,$ and P_Ω is the metric projection of H onto $\Omega.$ Then for all $u\in H$ and $v\in\Omega,$ we have $\|v-P_\Omega u\|^2+\|u-P_\Omega u\|^2\le\|u-v\|^2$.*

Lemma 1.7.

Bauschke and Combettes (2011)- Corollary 2.14: *Let H be a real Hilbert space. Then for all $\lambda\in[0,1]$ and $u,v\in H,$ we have*

$$\|\lambda u+(1-\lambda)v\|^2=\lambda\|u\|^2+(1-\lambda)\|v\|^2-\lambda(1-\lambda)\|u-v\|^2 .$$

Lemma 1.8.

Martinez-Yanes and Xu (2006) – Lemma 13: *Let H be a real Hilbert space and Ω be a nonempty, closed and convex subset of $H.$ Then for $x,y,z\in H$ and $a\in\mathbb{R},$ the following set is convex and closed: $\{w\in\Omega:\|y-w\|^2\le\|x-w\|^2+\langle z,w\rangle+a\}.$*

The following result will be used in the next section. The proof of this lemma is easy and is omitted.

Lemma 1.9.

Let H be a real Hilbert space. Then for all $u, v, w \in H,$ we have

$$\| u-v \|^2 = \| u-w \|^2 + \| w-v \|^2 + 2\langle u-w, w-v \rangle.$$

2. Main results

First, we denote by $F(T) = \{u \in H : Tu = u\}$ the set of fixed points of the mapping $T : H \to H.$ The following result is a sufficient condition for the closedness and convexity of the set $F(T)$ in real Hilbert spaces, where T is an asymptotically *G*-nonexpansive mapping.

Proposition 2.1.

Let H be a real Hilbert space, $G = (V(G), E(G))$ be a directed graph such that $V(G) = H,$ $T : H \to H$ be an asymptotically G-nonexpansive mapping with an asymptotic coefficient sequence $\{\lambda_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty} (\lambda_n - 1) < \infty,$ and $F(T) \times F(T) \subset E(G).$ Then

(1) If H have property $(G),$ then $F(T)$ is closed.

(2) If the graph G is transitive, $E(G)$ is coordinate-convex, then $F(T)$ is convex.

Proof.

(1). Suppose that $F(T) \neq \varnothing.$ Let $\{p_n\}$ be a sequence in $F(T)$ such that $\lim_{n\to\infty} \| p_n - p \| = 0$ for some $p \in H.$ Since $F(T) \times F(T) \subset E(G),$ we have $(p_n, p_{n+1}) \in E(G).$ By combining this with property (G) of $H,$ we conclude that there exists a subsequence $\{p_{n(k)}\}$ of $\{p_n\}$ such that $(p_{n(k)}, p) \in E(G)$ for $k \in \mathbb{N}.$ Since T is an asymptotically *G*-nonexpansive mapping, we obtain

$$\| p - Tp \| \le \| p - p_{n(k)} \| + \| Tp_{n(k)} - Tp \| \le (1 + \lambda_1) \| p - p_{n(k)} \|.$$

It follows from the above inequality and $\lim_{n\to\infty} \| p_n - p \| = 0$ that $Tp = p,$ that is, $p \in F(T).$ Therefore, $F(T)$ is closed.

(2). Let $p_1, p_2 \in F(T).$ For $t \in [0,1],$ we put $p = tp_1 + (1-t)p_2.$ Since $F(T) \times F(T) \subset E(G)$ and $p_1, p_2 \in F(T),$ we get $(p_1, p_1), (p_1, p_2), (p_2, p_1), (p_2, p_2) \in E(G).$ By combining this with $E(G)$ is coordinate-convex, we conclude that $t(p_1, p_1) + (1-t)(p_1, p_2) = (p_1, p) \in E(G),$ $t(p_1, p_1) + (1-t)(p_2, p_1) = (p, p_1) \in E(G)$ and $t(p_2, p_1) + (1-t)(p_2, p_2) = (p_2, p) \in E(G).$ Due to the fact that T is an asymptotically *G*-nonexpansive mapping, for each $i = 1, 2,$ we get

$$\| p_i - T^n p \| = \| T^n p_i - T^n p \| \le \lambda_n \| p_i - p \|. \tag{2.1}$$

Furthermore, by using Lemma 1.9, we get

$$\| p_1 - T^n p \|^2 = \| p_1 - p \|^2 + \| p - T^n p \|^2 + 2\langle p_1 - p, p - T^n p \rangle \tag{2.2}$$

and

$$\| p_2 - T^n p \|^2 = \| p_2 - p \|^2 + \| p - T^n p \|^2 + 2\langle p_2 - p, p - T^n p \rangle. \tag{2.3}$$

It follows from (2.1) and (2.2) that

$$\| p - T^n p \|^2 \le (\lambda_n^2 - 1) \| p_1 - p \|^2 - 2\langle p_1 - p, p - T^n p\rangle. \tag{2.4}$$

Also, we conclude from (2.1) and (2.3) that

$$\| p - T^n p \|^2 \le (\lambda_n^2 - 1) \| p_2 - p \|^2 - 2\langle p_2 - p, p - T^n p\rangle. \tag{2.5}$$

By multiplying t on the both sides of (2.4), and multiplying $(1-t)$ on the both sides of (2.5), we get

$$\begin{aligned} \| p - T^n p \|^2 &\le t(\lambda_n^2 - 1) \| p_1 - p \|^2 + (1-t)(\lambda_n^2 - 1) \| p_2 - p \|^2 \\ &\quad -2t\langle p_1 - p, p - T^n p\rangle - 2(1-t)\langle p_2 - p, p - T^n p\rangle \\ &= t(\lambda_n^2 - 1) \| p_1 - p \|^2 + (1-t)(\lambda_n^2 - 1) \| p_2 - p \|^2 . \end{aligned} \tag{2.6}$$

Since $\sum_{n=1}^{\infty}(\lambda_n - 1) < \infty$, we have $\lim_{n\to\infty} \lambda_n = 1$. Therefore, from (2.6), we find that

$$\lim_{n\to\infty} \| p - T^n p \| = 0. \tag{2.7}$$

Furthermore, since $(p_1, p) \in E(G)$ and T^n is edge-preserving, we have $(p_1, T^n p) \in E(G)$. Then, by the transitive property of *G* and $(p, p_1), (p_1, T^n p) \in E(G)$, we get $(p, T^n p) \in E(G)$. Due to asymptotically *G*-nonexpansiveness of T, we obtain

$$\| Tp - p \| \le \| Tp - T^{n+1} p \| + \| T^{n+1} p - p \| \le \lambda_1 \| p - T^n p \| + \| T^{n+1} p - p \| . \tag{2.8}$$

Taking the limit in (2.8) as $n \to \infty$ and using (2.7), we find that $Tp = p$, that is, $p \in F(T)$. Therefore, $F(T)$ is convex. □

Let $T_1, T_2 : H \to H$ be two asymptotically *G*-nonexpansive mappings with asymptotic coefficient sequences $\{\alpha_n\}, \{\beta_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(\alpha_n - 1) < \infty$ and $\sum_{n=1}^{\infty}(\beta_n - 1) < \infty$. Put $\lambda_n = \max\{\alpha_n, \beta_n\}$, we have $\{\lambda_n\} \subset [1, \infty)$ satisfying $\sum_{n=1}^{\infty}(\lambda_n - 1) < \infty$ and for all $(u, v) \in E(G)$ and for each $i = 1, 2,$ we have $\| T_i^n u - T_i^n v \| \le \lambda_n \| u - v \|$. In the following theorem, we also assume that $F = F(T_1) \cap F(T_2)$ is nonempty and bounded in H, that is, there exists a positive number κ such that $F \subset \{u \in H : \| u \| \le \kappa\}$. The following result shows the strong convergence of iteration process (1.5) to common fixed points of two asymptotically *G*-nonexpansive mappings in Hilbert spaces with directed graphs.

Theorem 2.2.

Let H be a real Hilbert space, H have property (G), $G = (V(G), E(G))$ be a directed transitive graph such that $V(G) = H$, $E(G)$ be coordinate-convex, $T_1, T_2 : H \to H$ be two asymptotically G-nonexpansive mappings such that $F(T_i) \times F(T_i) \subset E(G)$ for all $i = 1, 2,$ $\{u_n\}$ be a sequence generated by (1.5) where $\{a_n\}, \{b_n\}$ are sequences in [0,1] such that $0 < \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n < 1$, $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$; and $\gamma_n \in [\alpha, \beta]$ for some $\alpha, \beta \in \mathbb{R}$

such that $(u_n, p), (p, u_n), (z_n, p) \in E(G)$ *for all* $p \in F$; $\varepsilon_n = (\lambda_n^2 - 1)(1 + b_n\lambda_n^2)(\| z_n \| + \kappa)^2$. *Then the sequence* $\{u_n\}$ *strongly converges to* $P_F u_1$.

Proof.

The proof of Theorem 2.2 is divided into six steps.

Step 1. We show that $P_F u_1$ *is well-defined.* Indeed, by Proposition 2.1, we conclude that $F(T_1)$ and $F(T_2)$ are closed and convex. Therefore, $F = F(T_1) \cap F(T_2)$ is closed and convex. Note that F is nonempty by the assumption. This fact ensures that $P_F u_1$ is well-defined.

Step 2. We show that $P_{\Omega_{n+1}} u_1$ *is well-defined.* We first prove by a mathematical induction that Ω_n is closed and convex for $n \in \mathbb{N}$. Obviously, $\Omega_1 = H$ is closed and convex. Now we suppose that Ω_n is closed and convex. Then by the definition of Ω_{n+1} and Lemma 1.8, we conclude that Ω_{n+1} is closed and convex. Therefore, Ω_n is closed and convex for $n \in \mathbb{N}$.

Next, we show that $F \subset \Omega_{n+1}$ for all $n \in \mathbb{N}$. Indeed, for $p \in F$, we have $T_1 p = T_2 p = p$. Since $(z_n, p) \in E(G)$ and T_1^n is edge-preserving, we obtain $(T_1^n z_n, p) \in E(G)$. Due to the coordinate-convexity of $E(G)$, we get $(v_n, p) = (1 - b_n)(z_n, p) + b_n(T_1^n z_n, p) \in E(G)$. It follows from Lemma 1.7 and asymptotically G-nonexpansiveness of T_1, T_2 that

$$\begin{aligned}
\| w_n - p \|^2 &= \| (1 - a_n)(T_1^n v_n - p) + a_n(T_2^n v_n - p) \|^2 \\
&= (1 - a_n) \| T_1^n v_n - p \|^2 + a_n \| T_2^n v_n - p \|^2 - a_n(1 - a_n) \| T_2^n v_n - T_1^n v_n \|^2 \\
&\le (1 - a_n)\lambda_n^2 \| v_n - p \|^2 + a_n\lambda_n^2 \| v_n - p \|^2 - a_n(1 - a_n) \| T_2^n v_n - T_1^n v_n \|^2 \\
&= \lambda_n^2 \| v_n - p \|^2 - a_n(1 - a_n) \| T_2^n v_n - T_1^n v_n \|^2 \\
&\le \lambda_n^2 \| v_n - p \|^2
\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}
\| v_n - p \|^2 &= \| (1 - b_n)(z_n - p) + b_n(T_1^n z_n - p) \|^2 \\
&= (1 - b_n) \| z_n - p \|^2 + b_n \| T_1^n z_n - p \|^2 - b_n(1 - b_n) \| T_1^n z_n - z_n \|^2 \\
&\le (1 - b_n) \| z_n - p \|^2 + b_n\lambda_n^2 \| z_n - p \|^2 - b_n(1 - b_n) \| T_1^n z_n - z_n \|^2 \\
&\le [1 + b_n(\lambda_n^2 - 1)] \| z_n - p \|^2 - b_n(1 - b_n) \| T_1^n z_n - z_n \|^2 \\
&\le [1 + b_n(\lambda_n^2 - 1)] \| z_n - p \|^2 .
\end{aligned} \tag{2.10}$$

By substituting (2.10) into (2.9), we obtain

$$\begin{aligned}
\| w_n - p \|^2 &\le \lambda_n^2[1 + b_n(\lambda_n^2 - 1)] \| z_n - p \|^2 \\
&\le \| z_n - p \|^2 + (\lambda_n^2 - 1)(1 + b_n\lambda_n^2)(\| z_n \| + \| p \|)^2 \\
&\le \| z_n - p \|^2 + (\lambda_n^2 - 1)(1 + b_n\lambda_n^2)(\| z_n \| + \kappa)^2 \\
&= \| z_n - p \|^2 + \varepsilon_n.
\end{aligned} \tag{2.11}$$

It follows from (2.11) that $p \in \Omega_{n+1}$ and hence $F \subset \Omega_{n+1}$ for all $n \in \mathbb{N}$. Since $F \neq \varnothing$, we have $\Omega_{n+1} \neq \varnothing$ for all $n \in \mathbb{N}$. Therefore, we find that $P_{\Omega_{n+1}} u_1$ is well-defined.

Step 3. We show that $\lim_{n\to\infty} \| u_n - u_1 \|$ *exists.* Indeed, since $u_n = P_{\Omega_n} u_1$, we have

$\| u_n - u_1 \| \le \| x - u_1 \|$ for all $x \in \Omega_n$. (2.12)

Since $u_{n+1} = P_{\Omega_{n+1}} u_1 \in \Omega_{n+1} \subset \Omega_n$, by taking $x = u_{n+1}$ in (2.12), we obtain $\| u_n - u_1 \| \le \| u_{n+1} - u_1 \|$.

Since F is nonempty, closed and convex subset of H, there exists a unique $q = P_F u_1$ and hence $q \in F \subset \Omega_n$. Therefore, by choosing $x = q$ in (2.12), we get $\| u_n - u_1 \| \le \| q - u_1 \|$. By the above, we conclude that the sequence $\{ \| u_n - u_1 \| \}$ is bounded and nondecreasing. Therefore, $\lim_{n\to\infty} \| u_n - u_1 \|$ exists.

Step 4. We show that $\lim_{n\to\infty} u_n = u$ *for some* $u \in H$. Indeed, it follows from $u_n = P_{\Omega_n} u_1$ and Lemma 1.6, we get

$\| v - u_n \|^2 + \| u_1 - u_n \|^2 \le \| v - u_1 \|^2$ for all $v \in \Omega_n$. (2.13)

For $m > n$, we see that $u_m = P_{\Omega_m} u_1 \in \Omega_m \subset \Omega_n$. By taking $v = u_m$ in (2.13), we have $\| u_m - u_n \|^2 + \| u_1 - u_n \|^2 \le \| u_m - u_1 \|^2$. This implies that $\| u_m - u_n \|^2 \le \| u_m - u_1 \|^2 - \| u_n - u_1 \|^2$. It follows from the above inequality and the existence of $\lim_{n\to\infty} \| u_n - u_1 \|$ that $\lim_{m,n\to\infty} \| u_m - u_n \| = 0$ and hence $\{u_n\}$ is a Cauchy sequence. Therefore, there exists $u \in H$ such that $\lim_{n\to\infty} u_n = u$. Moreover, we also have

$\lim_{n\to\infty} \| u_{n+1} - u_n \| = 0.$ (2.14)

Step 5. We show that $u \in F$. Indeed, since $u_{n+1} \in \Omega_n$, by the definition of Ω_{n+1}, we get

$\| w_n - u_{n+1} \|^2 \le \| z_n - u_{n+1} \|^2 + \varepsilon_n$ (2.15)

It follows from $\| z_n - u_n \| = | \gamma_n | . \| u_n - u_{n-1} \|$ and (2.14) that

$\lim_{n\to\infty} \| z_n - u_n \| = 0.$ (2.16)

Therefore, we conclude from (2.14) and (2.16) that

$\lim_{n\to\infty} \| z_n - u_{n+1} \| = 0.$ (2.17)

It follows from (2.17) and the boundedness of the sequence $\{u_n\}$ that $\{z_n\}$ is bounded. Thus, there exists $A_1 > 0$ such that

$0 \le \varepsilon_n = (\lambda_n^2 - 1)(1 + b_n \lambda_n^2)(\| z_n \| + \kappa)^2 \le A_1(\lambda_n^2 - 1)$. Taking the limit in the above inequality as $n \to \infty$ and using $\lim_{n\to\infty} \lambda_n = 1$, we get $\lim_{n\to\infty} \varepsilon_n = 0$. Then, by combining this with (2.15) and (2.17), we have

$\lim_{n\to\infty} \| w_n - u_{n+1} \| = 0.$ (2.18)

It follows from (2.14) and (2.18) that

$\lim_{n\to\infty} \| w_n - u_n \| = 0.$ (2.19)

Then by combining (2.16) and (2.19), we obtain that

$$\lim_{n\to\infty} \| z_n - w_n \| = 0. \tag{2.20}$$

Next, for $p \in F,$ by the same proof of (2.9), (2.10) and (2.11), we get

$$\| w_n - p \|^2 \le \lambda_n^2[1 + b_n(\lambda_n^2 - 1)] \| z_n - p \|^2 - \lambda_n^2 b_n(1 - b_n) \| T_1^n z_n - z_n \|^2$$
$$\le \| z_n - p \|^2 + \varepsilon_n - b_n(1 - b_n) \| T_1^n z_n - z_n \|^2 . \tag{2.21}$$

It follows from (2.19) and the boundedness of the sequence $\{u_n\}$ that $\{w_n\}$ is bounded. Moreover, by the boundedness of $\{z_n\}$ and $\{w_n\}$, we conclude that there exists $A_2 > 0$ such that $\| z_n \| + \| w_n \| \le A_2$ for all $n \in \mathbb{N}.$ It follows from (2.21) that

$$b_n(1 - b_n) \| T_1^n z_n - z_n \|^2 \le \| z_n - p \|^2 - \| w_n - p \|^2 + \varepsilon_n$$
$$= \| z_n \|^2 - \| w_n \|^2 + 2\langle w_n - z_n, p\rangle + \varepsilon_n$$
$$\le (\| z_n \| - \| w_n \|)(\| z_n \| + \| w_n \|) + 2 \| w_n - z_n \| . \| p \| + \varepsilon_n$$
$$\le A_2 \| z_n - w_n \| + 2 \| w_n - z_n \| . \| p \| + \varepsilon_n. \tag{2.22}$$

Therefore, by combining (2.22) with (2.20) and using $\lim_{n\to\infty} \varepsilon_n = 0, \liminf_{n\to\infty} b_n(1 - b_n) > 0,$ we get

$$\lim_{n\to\infty} \| T_1^n z_n - z_n \| = 0. \tag{2.23}$$

Then by $(z_n, p), (p, u_n) \in E(G)$ and the transitive property of $G,$ we obtain $(z_n, u_n) \in E(G).$ Since T_1 is asymptotically *G*-nonexpansive and $(z_n, u_n) \in E(G),$ we get

$$\| T_1^n u_n - u_n \| \le \| T_1^n u_n - T_1^n z_n \| + \| T_1^n z_n - z_n \| + \| z_n - u_n \|$$
$$\le \lambda_n \| u_n - z_n \| + \| T_1^n z_n - z_n \| + \| z_n - u_n \|$$
$$= (1 + \lambda_n) \| z_n - u_n \| + \| T_1^n z_n - z_n \| . \tag{2.24}$$

It follows from (2.16), (2.23) and (2.24) that

$$\lim_{n\to\infty} \| T_1^n u_n - u_n \| = 0. \tag{2.25}$$

Next, by using similar argument as in the proof of (2.9), (2.10) and (2.11), we also obtain

$$\| w_n - p \|^2 \le \| z_n - p \|^2 + \varepsilon_n - a_n(1 - a_n) \| T_2^n v_n - T_1^n v_n \|^2 . \tag{2.26}$$

By the same proof of (2.22), from (2.26) and $\liminf_{n\to\infty} a_n(1 - a_n) > 0,$ we get

$$\lim_{n\to\infty} \| T_2^n v_n - T_1^n v_n \| = 0. \tag{2.27}$$

It follows from $\| v_n - z_n \| = b_n \| T_1^n z_n - z_n \|$ and (2.23) that

$$\lim_{n\to\infty} \| v_n - z_n \| = 0. \tag{2.28}$$

Then by combining (2.16) and (2.28), we have

$$\lim_{n\to\infty} \| u_n - v_n \| = 0. \tag{2.29}$$

Now, by $(v_n, p), (p, u_n) \in E(G)$ and the transitive property of $G,$ we obtain $(v_n, u_n) \in E(G).$ Since T_1, T_2 are asymptotically G-nonexpansive mappings, we get

$$\| T_2^n u_n - u_n \|$$
$$\le \| T_2^n u_n - T_2^n v_n \| + \| T_2^n v_n - T_1^n v_n \| + \| T_1^n v_n - T_1^n u_n \| + \| T_1^n u_n - u_n \|$$

$$\le \lambda_n \| v_n - u_n \| + \| T_2^n v_n - T_1^n v_n \| + \lambda_n \| v_n - u_n \| + \| T_1^n u_n - u_n \|. \tag{2.30}$$

It follows from (2.25), (2.27), (2.29) and (2.30) that

$$\lim_{n\to\infty} \| T_2^n u_n - u_n \| = 0. \tag{2.31}$$

Now, by combining $(u_n, p), (p, u_{n+1}) \in E(G)$ and the transitive property of G, we conclude that $(u_n, u_{n+1}) \in E(G)$. Then, for each $i = 1,2,$ due to the fact that T_i is an asymptotically *G*-nonexpansive mapping, we have

$$\| u_{n+1} - T_i^n u_{n+1} \| \le \| u_{n+1} - u_n \| + \| u_n - T_i^n u_n \| + \| T_i^n u_n - T_i^n u_{n+1} \|$$
$$\le \| u_{n+1} - u_n \| + \| u_n - T_i^n u_n \| + \lambda_n \| u_n - u_{n+1} \|$$
$$= (1+\lambda_n) \| u_{n+1} - u_n \| + \| u_n - T_i^n u_n \|. \tag{2.32}$$

It follows from (2.14), (2.25), (2.31) and (2.32) that

$$\lim_{n\to\infty} \| u_{n+1} - T_i^n u_{n+1} \| = 0. \tag{2.33}$$

Since $(p, u_{n+1}) \in E(G)$ for $p \in F$ and T_i^n is edge-preserving, we have $(p, T_i^n u_{n+1}) \in E(G)$. By combining this with $(u_{n+1}, p) \in E(G)$ and using the transitive property of $G,$ we obtain $(u_{n+1}, T_i^n u_{n+1}) \in E(G)$. Since T_i is an asymptotically *G*-nonexpansive mapping, we have

$$\| u_{n+1} - T_i u_{n+1} \| \le \| u_{n+1} - T_i^{n+1} u_{n+1} \| + \| T_i u_{n+1} - T_i^{n+1} u_{n+1} \|$$
$$\le \| u_{n+1} - T_i^{n+1} u_{n+1} \| + \lambda_1 \| u_{n+1} - T_i^n u_{n+1} \|. \tag{2.34}$$

Taking the limit in (2.34) as $n \to \infty$ and using (2.25), (2.31) and (2.33), we find that

$$\lim_{n\to\infty} \| T_i u_n - u_n \| = 0. \tag{2.35}$$

Therefore, by Lemma 1.5, (2.35), we find that $T_1 u = T_2 u = u$ and hence $u \in F$.

Step 6. We show that $u = q = P_F u_1$. Indeed, since $u_n = P_{\Omega_n} u_1$, we get

$$\langle u_1 - u_n, u_n - y \rangle \ge 0 \text{ for all } y \in \Omega_n. \tag{2.36}$$

Let $p \in F$. Since $F \subset \Omega_n$, we have $p \in \Omega_n$. Then, by choosing $y = p$ in (2.36), we obtain $\langle u_1 - u_n, u_n - p \rangle \ge 0$. Taking the limit in this inequality as $n \to \infty$ and using $\lim_{n\to\infty} u_n = u$, we find that $\langle u_1 - u, u - p \rangle \ge 0$. This implies that $u = P_F u_1$. □

Since every *G*-nonexpansive mapping is an asymptotically *G*-nonexpansive mapping with the asymptotic coefficient $\lambda_n = 1$ for all $n \in \mathbb{N}$, from Theorem 2.2, we get the following corollary.

Corollary 2.3.

Let H be a real Hilbert space, H have property (G), $G = (V(G), E(G))$ be a directed transitive graph such that $V(G) = H$, $E(G)$ be coordinate-convex, $T_1, T_2 : \Omega \to \Omega$ be two G-nonexpansive mappings such that $F = F(T_1) \cap F(T_2) \ne \varnothing$, $F(T_i) \times F(T_i) \subset E(G)$ for all $i = 1,2$, $\{u_n\}$ be a sequence generated by (1.4) where $\{a_n\}, \{b_n\}$ are sequences in [0,1] such that $0 < \liminf_{n\to\infty} a_n \le \limsup_{n\to\infty} a_n < 1$, $0 < \liminf_{n\to\infty} b_n \le \limsup_{n\to\infty} b_n < 1$; and $\gamma_n \in [\alpha, \beta]$ for some

$\alpha, \beta \in \mathbb{R}$ such that $(u_n, p), (p, u_n), (z_n, p) \in E(G)$ for all $p \in F$. Then the sequence $\{u_n\}$ strongly converges to $P_F u_1$.

Finally, we give a numerical example to illustrate for the convergence of the proposed iteration processes. In addition, the example also shows that the convergence of the proposed iteration processes to common fixed points of given mappings faster than some previous iteration processes.

Example 2.4.

Let $H = \mathbb{R}$, $G = (V(G), E(G))$ be a directed graph defined by $V(G) = H$, $E(G) = \{(u,v) : u, v \in [1, +\infty) \text{ and } u \neq v\} \cup \{(u,u) : u \in V(G)\}$. Then $E(G)$ is coordinate-convex and $\{(u,u) : u \in V(G)\} \subset E(G)$. Define three mappings $T_1, T_2, T_3 : H \to H$ by

$$T_1 u = \frac{1}{2}\sin^2(u-1) + 1, \; T_2 u = T_3 u = \frac{2u^2}{u^2+1} \text{ for all } u \in H.$$

Then, it is easy to check that T_1, T_2, T_3 are three asymptotically *G*-nonexpansive mappings with $\lambda_n = 1$ for all $n \in \mathbb{N}$. However, we see that $T_2 v - v = -\dfrac{v(v-1)^2}{v^2+1} \leq 0$ for all $v \geq 0$. This implies that $0 \leq T_2 v \leq v$ for all $v \geq 0$. Therefore, $0 \leq T_2^2 v = T_2(T_2 v) \leq T_2 v \leq v$. By continuing this process, we get that $0 \leq T_2^n v \leq T_2^{n-1} v \leq \ldots T_2 v \leq v$ for all $v \geq 0$ and $n \in \mathbb{N}$. By choosing $u = 1$ and $v = 0.7$, we obtain that $0 \leq T_2^n(0.7) \leq T_2(0.7) \leq 0.7$ for all $n \in \mathbb{N}$ and hence

$$|T_2^n u - T_2^n v| = |T_2^n(1) - T_2^n(0.7)| = 1 - T_2^n(0.7) \geq 1 - T_2(0.7) = \frac{51}{149} > 0.3 = |u - v|.$$

This implies that the condition $\|T_2^n u - T^n{}_2 v\| \leq \lambda_n \|u - v\|$ is not satisfied for $u = 1$, $v = 0.7$ and for all $\lambda_n \geq 1$. Therefore, T_2 is not an asymptotically nonexpansive mapping. Thus, some convergence results for asymptotically nonexpansive mappings can be not applicable to T_2. We also have $F = F(T_1) \cap F(T_2) \cap F(T_3) = \{1\} \neq \varnothing$. Consider

$$a_n = \frac{n+2}{4n+5}, b_n = \frac{n+1}{3n+7}, c_n = \frac{n+2}{8n+5} \text{ and } \gamma_n = \frac{n+1}{8n+3} \text{ for all } n \in \mathbb{N}.$$

By choosing $u_1 = 3$ and $u_2 = 2.5$. Then the numerical results of the iteration processes (1.1) – (1.5) are presented by the following table and figure.

Table 1. *Numerical results of the iteration processes (1.1) – (1.5)*

n	***Iteration (1.1)***	***Iteration (1.2)***	***Iteration (1.3)***	***Iteration (1.4)***	***Iteration (1.5)***
1	3.	3.	3.	3.	3.
2	2.3341045	2.1870207	2.2905893	2.5	2.5
3	1.812003	1.644584	1.8796074	1.9629608	1.8074613
4	1.4875208	1.3603696	1.5926171	1.5953041	1.3927414
…	…	…	…	…	…
27	1.0001777	1.000119	1.0000043	1.0000002	1.0000001
28	1.000132	1.0000884	1.0000025	1.0000001	1.
29	1.0000981	1.0000657	1.0000015	1.0000001	1.
30	1.0000729	1.0000488	1.0000009	1.	1.
…	…	…	…	…	…

35	1.0000166	1.0000111	1.0000001	1.	1.
36	1.0000124	1.0000083	1.	1.	1.
…	…	…	…	…	…
53	1.0000001	1.0000001	1.	1.	1.
54	1.0000001	1.	1.	1.	1.
55	1.	1.	1.	1.	1.

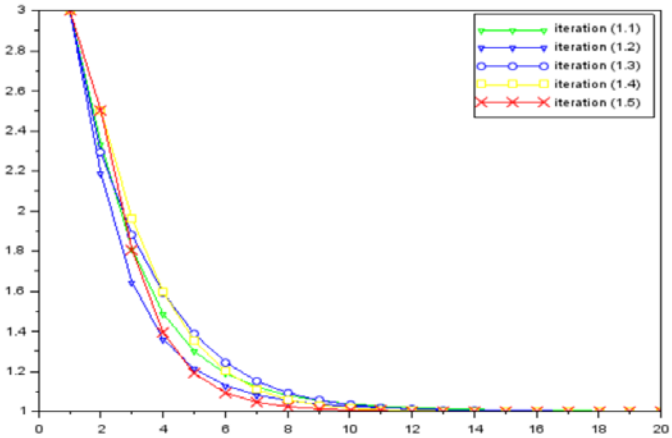

Figure 1. *Comparison of the convergence of iteration processes (1.1) – (1.5).*

Table 1 and Figure 1 show that for given mappings, the iteration processes (1.1) – (1.5) converge to 1. Furthermore, the convergence of the iteration process (1.5) to 1 is the fastest among other iteration processes. For the iteration processes for two *G*-nonexpansive mappings, the convergence of the iteration process (1.4) to 1 is faster than the iteration process (1.3). For the iteration processes for asymptotically *G*-nonexpansive mappings, the convergence of the iteration process (1.5) to 1 is faster than the iteration process (1.1) and (1.2).

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SỰ HỘI TỤ MẠNH CỦA DÃY LẶP LAI GHÉP CÓ YẾU TỐ QUÁN TÍNH CHO HAI ÁNH XẠ *G*-KHÔNG GIÃN TIỆM CẬN TRONG KHÔNG GIAN HILBERT VỚI ĐỒ THỊ

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TÓM TẮT

Trong bài báo này, bằng cách kết hợp phương pháp chiếu thu hẹp với dãy S-lặp cải tiến có yếu tố quán tính, chúng tôi giới thiệu một dãy lặp lai ghép có yếu tố quán tính cho hai ánh xạ G-không giãn tiệm cận và một dãy lặp lai ghép có yếu tố quán tính cho hai ánh xạ G-không giãn trong không gian Hilbert với đồ thị. Chúng tôi thiết lập điều kiện đủ cho tính lồi và đóng cho tập điểm bất động của ánh xạ G-không giãn tiệm cận trong không gian Hilbert với đồ thị. Sau đó, chúng tôi chứng minh định lí hội tụ mạnh cho việc tìm điểm bất động chung của hai ánh xạ G-không giãn tiệm cận trong không gian Hilbert với đồ thị. Từ định lí này, chúng tôi nhận được một kết quả hội tụ mạnh cho ánh xạ G-không giãn trong không gian Hilbert với đồ thị. Các kết quả này là sự mở rộng và tổng quát của một số kết quả hội tụ trong tài liệu tham khảo, trong đó giả thiết lồi của tập cạnh của đồ thị được thay bởi giả thiết lồi theo hướng. Đồng thời, chúng tôi cũng đưa ví dụ để minh họa cho sự hội tụ của những dãy lặp.

Từ khóa: ánh xạ *G*-không giãn tiệm cận; không gian Hilbert với đồ thị; dãy lặp lai ghép có yếu tố quán tính