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Email: tapchikhoahoc@hcmue.edu.vn; Website: http://tckh.hcmue.edu.vn

A NUMERICAL SCHEME FOR SOLUTIONS OF STOCHASTIC ADVECTION-DIFFUSION EQUATIONS

Nguyen Tien Dung**, Nguyen Anh Tra
Ho Chi Minh City University of Technology
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ABSTRACT

In this paper, finite difference schemes are proposed to approximate solutions of stochastic advection-diffusion equations. We used central-difference formula of third-order to approximate spatial derivatives. The stability, consistency and convergence of the scheme are analysed and established. A numerical result is also given to demonstrate the computational efficiency of the stochastic schemes.

Keywords: stochastic partial differential equation, finite difference method, convergence, stability.

TÓM TẮT

Một xấp xỉ nghiệm của phương trình khuếch tán bình lưu ngẫu nhiên

Trong bài báo này, phương pháp sai phân hữu hạn được sử dụng để xấp xỉ nghiệm của phương trình khuếch tán bình lưu ngẫu nhiên. Chúng tôi áp dụng công thức sai phân trung tâm bậc ba để ước lượng các đạo hàm riêng. Sự ổn định và sự hội tụ của lược đồ sai phân được nghiên cứu và đánh giá. Một ví dụ tính toán số cũng được xem xét để minh họa tính đúng đắn và hiệu quả của phương pháp xấp xỉ được đề xuất.

Từ khóa: phương trình đạo hàm riêng ngẫu nhiên, phương pháp sai phân hữu hạn, sự hội tụ, sự ổn định.

1. Introduction

Many applications in engineering and mathematical finance has developed with a heavy emphasis on stochastic partial differential equations (SPDEs). Apparently, appropriate algorithms that can approximate these equations have attracted many researchers since we can hardly find explicit formula of the corresponding solutions. In [2], [3], [4], [5], the authors studied the weak and the strong numerical schemes for SPDEs.

In this paper, we would like to propose a finite difference scheme for the following advection-diffusion

$$u_t(x,t) + \nu u_x(x,t) = \gamma u_{xx}(x,t) + \sigma u(x,t)\dot{W}(t) \tag{1}$$

* Email: *dungnt@hcmut.edu.vn*

with respect to an $\mathbb{R}^1$-valued Wiener process $(W(t), F_t)$ defined on a probability space (Ω, F, P), adapted to the standard filtration (F_t). The parameter γ is the viscosity coefficient and ν is the phase speed, and both are assumed to be positive. One may refers to [1] for applications of advection-diffusion equations in geophysics and [8] for its applications in consensus.

It is known that Young and Grygory [13] established an approximation scheme for one dimension advection-diffusion equation in 1973.
Later, [10], [11] proposed the idea of using three-point and five-point finite difference schemes to approximate the solution of stochastic diffusion equations without advection but unable to verify the corresponding stability and convergence. Similar approach using seven-point schemes is also implemented in [5] for the same equations. In 2011, [12] presented stochastic alternating direction explicit methods for advection-diffusion equations. In this paper, we would like to study the stability and convergence of a numerical scheme using three-point finite difference scheme for stochastic advection diffusion equations.

This paper is organised as follows: The next section introduces some preliminaries regarding to stochastic advection diffusion equation. In section 3, a three-point central difference scheme is presented and the stability and the convergence of the proposed scheme are carried out. Finally, the computational performance of the stochastic difference method is demonstrated in section 4.

2. Preliminaries

In this paper, we study a finite difference scheme for a stochastic advection diffusion equation

$$u_t(x,t) + \nu u_x(x,t) = \gamma u_{xx}(x,t) + \sigma u(x,t)\dot{W}(t), \text{ for all } t \in [0,T],\ x \in [0,l] \tag{2}$$

with initial-boundary conditions

$$\begin{aligned} u(x,0) &= u_0(x), \text{ for all } x \in [0,l] \\ u(0,t) &= f_0(t) \quad \text{and} \quad u(l,t) = f_l(t), \text{ for all } t \in [0,T] \end{aligned} \tag{3}$$

where $W(t)$ is a $\mathbb{R}^1$-valued Brownian motion, and ν, γ and σ are constants. One may refers to [12] for further discussions on the solutions of equations (2)-(3), including the existence and uniqueness.

For simplicity, we denote by L the following operator

$$Lu = u_t(x,t) + \nu u_x(x,t) - \gamma u_{xx}(x,t) - \sigma u(x,t)\dot{W}(t). \tag{4}$$

Then equation (2) becomes

$$Lu(x,t) = 0. \tag{5}$$

3. Three-point central difference scheme

In this section, we will apply three-point central difference formula to estimate the solution of (2).

Let Δx and Δt be the space step and the time step respectively such that $N=\dfrac{T}{\Delta t}$ and $K=\dfrac{l}{\Delta x}$ are positive integers.

Let $\lambda=\dfrac{\Delta t}{\Delta x}$ and $\rho=\dfrac{\Delta t}{(\Delta x)^2}$. For all $n=0,\ldots,N$, we denote

$$\begin{aligned}
u_k^{n+1} &= \big(1-2\gamma\rho\big)u_k^n+\Big(\gamma\rho-\frac{\nu\lambda}{2}\Big)u_{k-1}^n+\Big(\gamma\rho+\frac{\nu\lambda}{2}\Big)u_{k+1}^n \\
&\quad +\sigma u_k^n\Delta W_n,\quad k=1,\ldots,K-1 \\
u_0^{n+1} &= f_0((n+1)\Delta t) \\
u_K^{n+1} &= f_l((n+1)\Delta t) \\
u_k^0 &= u(k\Delta x,0),\quad k=0,\ldots,K.
\end{aligned} \tag{6}$$

where $\Delta W_n=W_{n+1}-W_n$ These equations give an approximation scheme for the solution of equations (2)-(3). For convenience, put $x_k=k\Delta x$ and $t_n=n\Delta t$, and we introduce the following operator

$$\begin{aligned}
L_k^n u_n = u_k^{n+1}-u_k^n+\nu\Delta t\left(\frac{u_{k+1}^n-u_{k-1}^n}{2\Delta x}\right)-\gamma\frac{\Delta t}{\Delta x^2}\left(u_{k-1}^n-2u_k^n+u_{k+1}^n\right) \\
-\sigma u_k^n[W(t_{n+1})-W(t_n)]
\end{aligned}$$

where $u_n=(u_0^n,\ldots,u_K^n)$ and $\bar{u}_n=[u(x_0,t_n),\ldots,u(x_K,t_n)]$.

We can then verify that (6) is equivalent to

$$\begin{aligned}
L_k^n u_n &= 0 \\
u_0 &= \bar{u}_0
\end{aligned}$$

We refer to [5] for the following definitions, but first we introduce for sequences $u=(\ldots,u_k,\ldots)$ the sup-norm $\|u_\infty\|=\sqrt{\sup_k|u_k|^2}$.

Definition 3.1.

A stochastic difference scheme $L_k^n u_n=0$ approximating the stochastic partial differential equation $Lv=0$ is convergent in mean square at time t if, as $\Delta x\to 0$

$$\mathrm{E}\left\|u^N-v^N\right\|_\infty^2\to 0$$

where $u^N=(\ldots,u_k^N,\ldots)$ and $v^N=(\ldots,v_k^N,\ldots)$.

Definition 3.2.

A stochastic difference scheme is said to be stable with respect to a norm in mean square if there exist positive constants $\overline{\Delta x_0}$ *and* $\overline{\Delta t_0}$, *and nonnegative constants* K *and* β *such that*

$$E\,\|u^N\|^2 \le Ke^{\beta T}E\,\|u^0\|^2,$$

for all $0 \le \Delta x \le \overline{\Delta x_0}$ *and* $0 \le \Delta t \le \overline{\Delta t_0}$.

In what follows, we will study the consistence, the stability and the convergence of scheme (6). For convenience, we use notation $\|\cdot\|_\infty$ to denote the supremum norm.

Theorem 3.3.

If $\frac{\nu\lambda}{2} \le \gamma\rho \le \frac{1}{2}$, *then scheme (6) with a fixed space step* Δx *is conditionally stable. In fact, there exists a constant* C *such that*

$$\sup_k \mathrm{E}\,|u_k^n|^2 \le C \sup_k \mathrm{E}\,|u_k^0|^2 \quad \textit{for all } n \ge 0.$$

Proof. Equation (6) implies that

$$\mathrm{E}\,|u_k^{n+1}|^2 = \mathrm{E}\left|\left(1-2\gamma\rho\right)u_k^n + \left(\gamma\rho - \frac{\nu\lambda}{2}\right)u_{k-1}^n + \left(\gamma\rho + \frac{\nu\lambda}{2}\right)u_{k+1}^n\right|^2 + \mathrm{E}(\sigma^2)(\Delta t)\mathrm{E}\,|u_k^n|^2 \qquad (7)$$

If $\gamma\rho \ge \frac{\nu\lambda}{2}$, then (7) becomes

$$\mathrm{E}\,|u_k^{n+1}|^2 \le \mathrm{E}\left[1+\sigma^2\Delta t\right] \sup_{k=0,\dots,K} \mathrm{E}\,|u_k^n|^2$$

Thus

$$\sup_{k=0,\dots,K} \mathrm{E}\,|u_k^{n+1}|^2 \le (1+\sigma^2\Delta t) \sup_{k=0,\dots,K} \mathrm{E}\,|u_k^n|^2$$

for all $n \ge 0$. Consequently,

$$\begin{aligned} \sup_{k=0,\dots,K} \mathrm{E}\,|u_k^n|^2 &\le (1+\sigma^2\Delta t)^n \sup_{k=0,\dots,K} \mathrm{E}\,|u_k^0|^2 \\ &\le e^{\sigma^2 T} \sup_{k=0,\dots,K} \mathrm{E}\,|u_k^0|^2 \end{aligned} \qquad (8)$$

□

Theorem 3.4.

If $\frac{\nu\lambda}{2} \le \gamma\rho \le \frac{1}{2}$ *then scheme* (6) *converges in norm* $\|\cdot\|_\infty$ *to the solution of equations* (2)-(3).

Proof. First of all, (6) implies that

$$\begin{aligned} u_k^{n+1} = \ & u_k^n + \gamma \frac{\Delta t}{\Delta x^2}\left(u_{k-1}^n - 2u_k^n + u_{k+1}^n\right) - \nu \Delta t \frac{u_{k+1}^n - u_{k-1}^n}{2\Delta x} \\ & + \sigma u_k^n \left(W((n+1)\Delta t) - W(n\Delta t)\right). \end{aligned} \tag{9}$$

On the other hand, denote by v_k^n the value of the solution of equation (2) at (x_k, t_n). Assume that $s \in [t_n, t_{n+1}]$. We have

$$\begin{aligned} v_x(x_k, s) \ &= \frac{v_{k+1}^n - v_{k-1}^n}{2\Delta x} + \frac{\Delta t}{2\Delta x}\left[v_t(\mathrm{x}_{k+1}, \mathrm{t}_n + \zeta_{k+1}(\mathrm{s})\Delta\, \mathrm{t}) - v_t(\mathrm{x}_{k-1}, \mathrm{t}_n + \zeta_{k-1}(\mathrm{s})\Delta\, \mathrm{t})\right] \\ & - \frac{(\Delta x)^2}{12}\left[v_{xxx}(x_k + \theta_{k+1}(s)\Delta x, s) + v_{xxx}(x_k - \theta_{k-1}(s)\Delta x, s)\right] \end{aligned} \tag{10}$$

where $0 \le \theta_{k-1}(r), \theta_{k+1}(r) \le 1$. Similarly

$$\begin{aligned} v_{xx}(x_k, s) \ &= \frac{1}{(\Delta x)^2}\left[v_{k-1}^n - 2v_k^n + v_{k+1}^n\right] + \frac{\Delta t}{(\Delta x)^2}[v_t(\mathrm{x}_{k-1}, \mathrm{t}_n + \zeta_{k-1}(\mathrm{s})\Delta\, \mathrm{t}) \\ & -2v_t(\mathrm{x}_k, \mathrm{t}_n + \zeta_k(\mathrm{s})\Delta\, \mathrm{t}) + v_t(\mathrm{x}_{k+1}, \mathrm{t}_n + \zeta_{k+1}(\mathrm{s})\Delta\, \mathrm{t})] \\ & - \frac{\Delta x}{6}\left[v_{xxxx}(x_k + \theta_{k+1}(\mathrm{s})\Delta x, \mathrm{s}) + v_{xxxx}(x_k - \theta_{k-1}(\mathrm{s})\Delta x, \mathrm{s})\right] \end{aligned} \tag{11}$$

where $0 \le \zeta_{k-1}(\mathrm{s}), \zeta_k(\mathrm{s}), \zeta_{k+1}(\mathrm{s}) \le 1$. For the sake of simplicity, we denote

$$\begin{aligned} \psi_{k+1}^{+}(\mathrm{s}) \ &= v_{xxx}(x_k + \theta_{k+1}(\mathrm{s})\Delta x, \mathrm{s}) \\ \psi_{k-1}^{-}(\mathrm{s}) \ &= v_{xxx}(x_k - \theta_{k-1}(\mathrm{s})\Delta x, \mathrm{s}) \end{aligned}$$

and

$$\phi_{k+i}(\mathrm{s}) \ = v_t(x_{k+i}, t_n + \zeta_{k+i}(\mathrm{s})\Delta t)$$

for all $i = -1, 0, 1$. Integrating both sides of equation (2) from t_n to t_{n+1}, and then substituting v_x and v_{xx} given by equations (10) and (11) into the resulting equation, we deduce

$$\begin{aligned} v_k^{n+1} &= v_k^n - \nu\int_{t_n}^{t_{n+1}} v_x(x_k,s)ds + \gamma\int_{t_n}^{t_{n+1}} v_{xx}(x_k,s)ds + \sigma\int_{t_n}^{t_{n+1}} v(x_k,s)dW(s) \\ &= v_k^n - \nu\int_{t_n}^{t_{n+1}} [\frac{v_{k+1}^n - v_{k-1}^n}{2\Delta x} + \frac{\Delta t}{2\Delta x}(\phi_{k+1}(\mathrm{s}) - \phi_{k-1}(\mathrm{s})) \\ &\quad -\nu\frac{(\Delta x)^2}{12}\left(\psi_{k+1}^{+}(s) + \psi_{k-1}^{-}(s)\right)\Big]ds \\ &\quad +\gamma\int_{t_n}^{t_{n+1}} [\frac{1}{(\Delta x)^2}(v_{k+1}^n - 2v_k^n + v_{k-1}^n) \\ &\quad +\frac{\Delta t}{(\Delta x)^2}[(\phi_{k-1}(\mathrm{s}) - 2\phi_k(\mathrm{s}) + \phi_{k-1}(\mathrm{s})] \\ &\quad -\gamma\frac{\Delta x}{6}(\psi_{k+1}^{+}(\mathrm{s}) - \psi_{k-1}^{-}(\mathrm{s}))] + \sigma\int_{t_n}^{t_{n+1}} v(\mathrm{x}_k,\mathrm{s})dW(\mathrm{s}) \end{aligned} \tag{12}$$

Put $z_k^n = v_k^n - u_k^n$ and $z^n = (z_0^n, \ldots, z_K^n)$. We can derive from (9) and (12) that for all $k = 1, \ldots, K-1$

$$\begin{aligned} z_k^{n+1} &= (1-2\gamma\rho)z_k^n + (\gamma\rho - \frac{\nu\lambda}{2})z_{k-1}^n + (\gamma\rho + \frac{\nu\lambda}{2})z_{k+1}^n \\ &-\nu\int_{t_n}^{t_{n+1}}\Big[\frac{\Delta t}{2\Delta x}\left(\phi_{k+1}(s) - \phi_{k-1}(s)\right) - \frac{(\Delta x)^2}{12}\left(\psi_{k+1}^{+}(s) + \psi_{k-1}^{-}(s)\right)\Big]ds \\ &+\gamma\int_{t_n}^{t_{n+1}}\Big[\frac{\Delta t}{(\Delta x)^2}\left(\phi_{k-1}(s) - 2\phi_k(s) + \phi_{k+1}(s)\right) \\ &-\frac{\Delta x}{6}\left(\psi_{k+1}^{+}(s) + \psi_{k-1}^{-}(s)\right)\Big] + \sigma\int_{t_n}^{t_{n+1}}(v(x_k,s) - u_k^n)dW(s). \end{aligned} \tag{13}$$

If $\gamma\rho \ge \dfrac{\nu\lambda}{2}$ then

$$\begin{aligned} &\left|(1-2\gamma\rho)z_k^n + (\gamma\rho - \frac{\nu\lambda}{2})z_{k-1}^n + (\gamma\rho + \frac{\nu\lambda}{2})z_{k+1}^n\right| \\ &\le \Big[(1-2\gamma\rho) + (\gamma\rho - \frac{\nu\lambda}{2}) + (\gamma\rho + \frac{\nu\lambda}{2})\Big]\sup_{k=1,\ldots,K-1}|z_k^n| \\ &= \sup_{k=1,\ldots,K-1}|z_k^n| \end{aligned} \tag{14}$$

Besides, for any given $\delta_1 > 0$ and real numbers a and b.

$$(a+b)^2 \le ca^2 + \frac{c}{c-1}b^2 \tag{15}$$

where $c = 1 + \delta_1\Delta t > 1$.

It can be derived from equation (13) to (15) that

$$
\begin{aligned}
\mathrm{E}\,|z_k^{n+1}|^2 \quad &\le (1+\delta_1\Delta t)\,\mathrm{E}\Big|(1-2\gamma\rho)z_k^n + (\gamma\rho-\frac{\nu\lambda}{2})z_{k-1}^n + (\gamma\rho+\frac{\nu\lambda}{2})z_{k+1}^n \\
&+\sigma\int_{t_n}^{t_{n+1}}(v(x_k,s)-u_k^n)dW(s)\Big|^2 \\
&+\frac{1+\delta_1\Delta t}{\delta_1\Delta t}\,\mathrm{E}\Big|-\nu\int_{t_n}^{t_{n+1}}\Big[\frac{\Delta t}{2\Delta x}\big(\phi_{k+1}(s)-\phi_{k-1}(s)\big) \\
&-\frac{(\Delta x)^2}{12}\big(\psi_{k+1}^{+}(s)+\psi_{k-1}^{-}(s)\big)\Big]ds \\
&+\gamma\int_{t_n}^{t_{n+1}}\Big[\frac{\Delta t}{(\Delta x)^2}\big(\phi_{k-1}(s)-2\phi_k(s)+\phi_{k+1}(s)\big) \\
&-\frac{\Delta x}{6}\big(\psi_{k+1}^{+}(s)+\psi_{k-1}^{-}(s)\big)\Big]ds\Big|^2 \le (1+\delta_1\Delta t)\sup_{k=1,\ldots,K-1}\mathrm{E}\,|z_k^n|^2 \\
&+(1+\delta_1\Delta t)\mathrm{E}(\sigma^2)\times\sup_{k=1,\ldots,K-1}\int_{t_n}^{t_{n+1}}\mathrm{E}\,|v(x_k,s)-v_k^n|^2\,ds \\
&+(1+\delta_1\Delta t)\mathrm{E}(\sigma^2)\sup_{k=1,\ldots,K-1}\int_{t_n}^{t_{n+1}}\mathrm{E}\,|z_k^n|^2\,ds \\
&+\frac{1+\delta_1\Delta t}{\delta_1\Delta t}K(\Delta t)^2\big[\lambda^2+(\Delta x)^2\big] \\
&\le (1+\delta_1\Delta t)\big(1+\mathrm{E}(\sigma^2)\Delta t\big)\sup_{k=1,\ldots,K-1}\mathrm{E}\,|z_k^n|^2 + K\Delta t\big[\lambda^2+(\Delta x)^2\big]
\end{aligned}
$$

We choose $\delta_1 \ge \mathrm{E}(\sigma^2)$. Then for all k and n

$$\mathrm{E}\,|z_k^{n+1}|^2 \le (1+\delta_1\Delta t)^2\sup_{k=1,\ldots,K-1}\mathrm{E}\,|z_k^n|^2 + K\Delta t\big[\lambda^2+(\Delta x)^2\big]$$

which implies that

$$\mathrm{E}\,\|z^{n+1}\|_\infty^2 \le (1+\delta_1\Delta t)^2\,\mathrm{E}\,\|z^n\|_\infty^2 + K\Delta t\big[\lambda^2+(\Delta x)^2\big] \tag{16}$$

where $z^n = (\cdots, z_k^n, \cdots)$. Since $z^0 = 0$, it follows that

$$
\begin{aligned}
E\,\|z^n\|^2 \quad &\le (1+\delta_1\Delta t)^{2n}E\,\|z^0\|^2 + K\Delta t\big[\lambda^2+(\Delta x)^2\big]\sum_{j=0}^{n-1}(1+\delta_1\Delta t)^{2j} \\
&\le K\big[\lambda^2+(\Delta x)^2\big]\frac{(1+\delta_1\Delta t)^{2n}-1}{2\delta_1+\delta_1^2\Delta t} \\
&\le K\big[\lambda^2+(\Delta x)^2\big]\frac{e^{2\delta_1 T}-1}{2\delta_1+\delta_1^2\Delta t}.
\end{aligned}
$$

whose the right-hand side decays to 0 as both Δx and $\frac{(\Delta t)^2}{\Delta x}$ approach 0. This completes the proof of this theorem. □

4. Numerical results

In this section, the performance of the presented numerical techniques described in the previous sections for solving the proposed SPDEs is considered and applied to a test problem. For computational purposes, it is useful to consider discretised Brownian motion where $W(t)$ is specified at discrete t values.

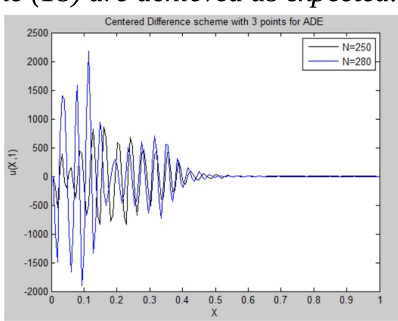
Example 4.1. *Let us consider the following advection diffusion equation*

$$\begin{aligned} &u_t(x,t) = \gamma u_{xx}(x,t) + \nu u_x(x,t) + \sigma u(x,t)dW(t), \text{ for all } t \in [0,1], x \in [0,1] \\ &u(x,0) = x^2(1-x)^2, \text{ for all } x \in [0,1] \\ &u(0,t) = u(1,t) = 0 \end{aligned} \tag{17}$$

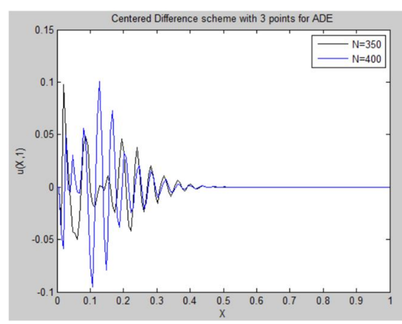
where $\gamma = 0.001$, $\nu = \sigma = 1$, *and* $W(t)$ *is Brown motion. We will use algorithm (6) to approximate the solution of equation (17) as follows*

$$u_k^{n+1} = \left(1-2\gamma\rho\right)u_k^n + \left(\gamma\rho - \frac{\nu\lambda}{2}\right)u_{k-1}^n + \left(\gamma\rho + \frac{\nu\lambda}{2}\right)u_{k+1}^n + \sigma u_k^n \Delta W_n. \tag{18}$$

Assume that $\Delta t = \dfrac{1}{N}$ *and* $\Delta x = \dfrac{1}{M}$. *As stated in theorems Theorem 3.3 and Theorem 3.44, the sufficient condition for the stability and the convergence of scheme (18) is* $\gamma\rho \le \dfrac{1}{2}$. *If* $M = 150$ *then we need* $N \ge 45$. ***Figure 1*** *shows that the stability and the convergence of scheme (18) are achieved as expected.*



(a) N = 250 v N = 280



(b) N = 350 v N = 400

Figure 1. *Approximation of* $u(x;1)$ *with different* N

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