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Research Article

A WEIGHTED APPROACH FOR CACCIOPOLI INEQUALITY FOR SOLUTIONS TO *P*-LAPLACE EQUATIONS

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ABSTRACT

Weighted fractional Sobolev spaces have many applications in partial differential equations. In this paper, we study a class of weighted fractional Sobolev spaces, where the weights are the distance functions to the boundary of the defined domain. This class has been used to obtain a weighted Cacciopoli-type inequality for solutions to p-Laplace equations with measure data. Our result expands to the Cacciopoli inequality in a recent paper by Tran and Nguyen (2021b).

Keywords: Cacciopoli-type inequality; partial differential equations; *p*-Laplace equations; weighted fractional Sobolev spaces

1. Introduction

In this paper, we are interested in the following Dirichlet problem with measure data

$$\begin{cases} -\operatorname{div}\left(\mathcal{A}\left(x,\nabla u\right)\right) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where the domain $\Omega \subset \mathbb{R}^n$ is open and bounded, and the given data μ is a Borel measure with finite mass in Ω. The operator $\mathcal{A}$ is close to the operator $\xi \mapsto |\xi|^{p-2}\xi,\ \xi \in \mathbb{R}^n$, this means

$$g_1\left(|\xi|\right)\mathrm{Id}_n \le \partial_\xi \mathcal{A}\left(\bullet,\xi\right) \le g_2\left(|\xi|\right)\mathrm{Id}_n,$$

where $g_1\left(|\xi|\right) \approx g_2\left(|\xi|\right) \approx |\xi|^{p-2}$. It is well-known that when $p = 2$, if the data μ belongs to the Lebesgue space $L^q_{\mathrm{loc}}\left(\Omega\right)$ then ∇u belongs to the Sobolev space $W^{1,q}_{\mathrm{loc}}\left(\Omega\right)$:

$$\mu \in L^q_{\mathrm{loc}}\left(\Omega\right) \Rightarrow \nabla u \in W^{1,q}_{\mathrm{loc}}\left(\Omega\right),\ 1 < q < \infty. \tag{1.2}$$

We hope that (1.2) still true for $q = 1$, but instance, in the recent paper by Avelin *et al.* (2018), the authors showed that the result just holds for the fractional Sobolev spaces. More precisely, they proved that

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$$\mu \in L^1_{\text{loc}}(\Omega) \;\Rightarrow\; \nabla u \in W^{\sigma,1}_{\text{loc}}(\Omega),\; 0<\sigma<1. \tag{1.3}$$

Moreover, also in the same paper, the authors gave a very important regularity result when $2-\frac{1}{n}<p\le 2$. Let us recall the following theorem:

Theorem 1.1. (Avelin, Kuusi & Mingione, 2018) *Let* Ω *be an open subset of* $\mathbb{R}^n$ *and* $p>2-\frac{1}{n}$. *Assume that* $u \in W^{1,\max\{1,p-1\}}_{\text{loc}}(\Omega)$ *is a SOLA solution to* (1.1). *Then for any* $\sigma \in (0,1)$ *one has*

$$\mathcal{A}(\nabla u) \in W^{\sigma,1}_{\text{loc}}(\Omega). \tag{1.4}$$

Moreover, there exists a constant $C = C(c_{\mathcal{A}},\sigma,n,p)>0$ *such that*

$$\frac{1}{|\mu|(B_{R/2})}\int_{B_{R/2}}\int_{B_{R/2}}\frac{\left|\mathcal{A}(\nabla u(x))-\mathcal{A}(\nabla u(y))\right|}{|x-y|^{n+\sigma}}dxdy \le \frac{C}{R^{\sigma}}\left(\frac{1}{|\mu|(B_R)}\int_{B_R}\left|\mathcal{A}(\nabla u(x))\right|dx+\left[\frac{|\mu|(B_R)}{R^{n-1}}\right]\right), \tag{1.5}$$

for every ball $B_R \Subset \Omega$.

We remark that the weak solution to the measure data problem (1.1) may be not unique. To ensure the existence and uniqueness of the solution to (1.1), we deal with the SOLA solution which has been defined in Benilan et al. (1995) and Maso et al. (1999). There are interesting results related to regularity for solutions to the measure data problem (1.1), such as (Mingione, 2007), (Tran & Nguyen, 2019, 2020a, 2021a), (Balci et al., 2020), etc.

Recently, Tran and Nguyen established the global regularity result of (1.4) in Tran and Nguyen (2021). However, they only proved that $\mathcal{A}(\nabla u)$ belongs to the weighted fractional Sobolev space, even for the smooth domain Ω. In the present article, we improve the result reported by Tran and Nguyen (2021) by proving the inequality similar to (1.5), where the weights are both on the left-hand and right-hand side. In other words, we prove the following inequality

$$\int_{\Omega}\int_{\Omega} d^{\alpha}(x)d^{\beta}(y)\frac{\left|\mathcal{A}(\nabla u(x))-\mathcal{A}(\nabla u(y))\right|}{|x-y|^{n+\sigma}}dxdy \le C\left(\int_{\Omega} d^{\gamma}(x)\left|\mathcal{A}(\nabla u(x))\right|dx+|\mu|(\Omega)\right), \tag{1.6}$$

where $d(x) := \text{dist}(x,\partial\Omega)$ defines the distance from x to the boundary of the domain. Here the result holds for every α, $\beta>0$ and $\gamma \ge 0$ satisfying $\alpha>\gamma$, $\beta>\gamma$, $\alpha+\beta-\gamma>\sigma$.

Motivated by these works, we first consider some basic properties of the weighted fractional Sobolev spaces, in which the weights are the power of distances to the boundary.

Then we prove the weighted Cacciopoli type inequality (1.6) which corresponds to the SOLA solution to the measure data problem (1.1).

The rest of the article will be organized as follows. In the next section, we introduce the weighted fractional Sobolev spaces by introducing some basic notation, definitions, and some properties of weighted fractional Sobolev spaces. Then, we end up with a section that introduces the main results and proving the main results in this paper, and it allows us to conclude a weighted approach for Cacciopoli inequality for solutions to *p*-Laplace equations (1.1).

2. Preliminaries

2.1. Basic notation

In this article, the constant depends on real numbers α , β , and γ will be denoted by $C(\alpha,\beta,\gamma)$. From now on, $B_\rho(\zeta)$ stands for the ball with radius ρ and centered at $\zeta\in\Omega$. Finally, for $1\le p<\infty$, we will denote by $L^p(\Omega)$ the usual Lebesgue spaces; and the Sobolev spaces is signed as $W^{s,p}(\Omega)$.

2.2. Fractional Sobolev spaces

We now introduce the definition of fractional Sobolev spaces, see (Avelin, Kuusi & Mingione, 2018) and (Di Nezza, Palatucci & Valdinoci, 2012) for instance.

Definition 2.1. *(The fractional Sobolev space) Assume that $\Omega\subset\mathbb{R}^n$ is an open set with $n\ge 2$, s is the fraction in $(0,1)$ and $p\in[1,+\infty)$. Then, the fractional Sobolev space $W_G^{s,p}(\Omega)$ is defined as follows*

$$W_G^{s,p}(\Omega):=\left\{u\in L^p(\Omega):\ \frac{|u(x)-u(y)|}{|x-y|^{\frac{n}{p}+s}}\in L^p(\Omega\times\Omega)\right\}, \tag{2.1}$$

with the natural norm

$$\|u\|_{W_G^{s,p}(\Omega)}=\left[\int_\Omega |u(x)|^p\,dx+\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy\right]^{\frac{1}{p}}. \tag{2.2}$$

The Gagliardo semi-norm of u is defined by

$$[u]_{W_G^{s,p}(\Omega)}:=\left[\int_\Omega\int_\Omega \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}}dxdy\right]^{\frac{1}{p}}. \tag{2.3}$$

Furthermore, we defined $W_{G,\mathrm{loc}}^{\sigma,1}(\Omega)$ as

$$W_{G,\mathrm{loc}}^{\sigma,1}(\Omega):=\left\{v\in W_G^{\sigma,1}(\Omega_1):\forall\Omega_1\subset\Omega,\Omega_1\ \textit{is compact}\right\}. \tag{2.4}$$

Let us introduce some properties of weighted fractional Sobolev spaces

Lemma 2.2. *Assume that $\Omega\subset\mathbb{R}^n$ is an open domain, $p\in[1,+\infty)$ and $u:\Omega\to\mathbb{R}$ is a measurable function. Then*

$$\|u\|_{W_G^{s,p}(\Omega)} \le \|u\|_{W_G^{t,p}(\Omega)}, \text{ for all } t \in (s,1).$$

It follows that

$$W_G^{t,p}(\Omega) \subseteq W_G^{s,p}(\Omega), \text{ for all } t \in (s,1).$$

If we have Ω is the bounded Lipschitz domain, then we have the following lemma.

Lemma 2.3. *Assume that $\Omega \subset \mathbb{R}^n$ is an open bounded and Lipschitz domain, $p \in [1,+\infty)$ and $u:\Omega \to \mathbb{R}$ is a measurable function. Then*

$$W_G^{1,p}(\Omega) \subseteq W_G^{s,p}(\Omega), \text{ for all } s \in (0,1).$$

Proof of Lemma 2.2 and Lemma 2.3 can be found in a study by Di Nezza, Palatucci, and Valdinoci(2012).

2.3. Weighted fractional Sobolev spaces

Since the main content of the article uses some properties of weighted fractional Sobolev space where the weights are the distance functions to the boundary of the domain. We will introduce *weighted fractional Sobolev spaces* via the following definition.

Definition 2.4. *(Weighted fractional Sobolev space) Assume that $\Omega \subset \mathbb{R}^n$ is an open, bounded and Lipschitz domain, $q \in [1,\infty)$, $s \in (0,1)$ and α, $\beta \ge 0$. Then, we define the weighted fractional Sobolev space as*

$$W_G^{s,p}(\Omega;\alpha,\beta) := \left\{ u \in L^p(\Omega) : \int_\Omega \int_\Omega d^\alpha(x) d^\beta(y) \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy < \infty \right\}, \tag{2.5}$$

with the natural norm

$$\|u\|_{W_G^{s,p}(\Omega;\alpha,\beta)} = \left[\int_\Omega |u(x)|^p dx + \int_\Omega \int_\Omega d^\alpha(x) d^\beta(y) \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right]^{\frac{1}{p}}. \tag{2.6}$$

where $d(x) = \operatorname{dist}(x,\partial\Omega)$.

Similar to the non-weight spaces, the weighted Gagliardo semi-norm of $W_G^{s,p}(\Omega;\alpha,\beta)$ is defined by

$$[u]_{W_G^{s,p}(\Omega;\alpha,\beta)} := \left[\int_\Omega \int_\Omega d^\alpha(x) d^\beta(y) \frac{|u(x)-u(y)|^p}{|x-y|^{n+sp}} dxdy \right]^{\frac{1}{p}}. \tag{2.7}$$

Let us introduce some properties of weighted fractional Sobolev space, which is similar to fractional Sobolev space.

Lemma 2.5. *Assume that $u:\Omega \to \mathbb{R}$ is a measurable function. Then, there exists a constant $C \ge 1$ such that*

$$\|u\|_{W_G^{s,p}(\Omega;\alpha,\beta)} \le C \|u\|_{W_G^{t,p}(\Omega;\alpha,\beta)}, \text{ for all } t \in (s,1).$$

In particular,

$$W_G^{t,p}(\Omega;\alpha,\beta) \subseteq W_G^{s,p}(\Omega;\alpha,\beta), \quad \text{for all } t \in (s,1).$$

The proof is similar in spirit to the proof of Lemma 2.2. Now, we establish the connection between fractional Sobolev space and weighted fractional Sobolev space by the following lemma.

Lemma 2.6. *For every $\alpha, \beta \geq 0$ we have*

$$[u]_{W_G^{s,p}(\Omega;\alpha,\beta)} \leq \left(\mathrm{diam}(\Omega)\right)^{\frac{\alpha+\beta}{q}} [u]_{W_G^{s,p}(\Omega)},$$

and it yields

$$W_G^{s,p}(\Omega) \subset W_G^{s,p}(\Omega;\alpha,\beta).$$

That means that weighted fractional Sobolev space is the expansion of fractional Sobolev space, and the result we have obtained is more general. In the following section, we introduce the main results and prove the main results.

3. Main results

In this section, we state our main results and their proofs.

Theorem 3.1. *Let $p > 2 - \frac{1}{n}$, $\sigma \in (0,1)$ and Ω be an open bounded and smooth domain in $\mathbb{R}^n$. Assume that $u \in W^{1,\max\{1,p-1\}}(\Omega)$ is a SOLA solution to* (1.1). *Then for every α, $\beta > 0$ and $\gamma \geq 0$ satisfying $\alpha > \gamma$, $\beta > \gamma$, $\alpha + \beta - \gamma > \sigma$, there exists a constant $C > 0$ such that*

$$\int_\Omega\int_\Omega d^\alpha(x) d^\beta(y) \frac{\left|\mathcal{A}(\nabla u(x)) - \mathcal{A}(\nabla u(y))\right|}{|x-y|^{n+\sigma}} dxdy \leq C\left(\int_\Omega d^\gamma(x)\left|\mathcal{A}(\nabla u(x))\right| dx + |\mu|(\Omega)\right), \tag{3.1}$$

where $d^\theta(x) := [\mathrm{dist}(x,\partial\Omega)]^\theta$.

In this section, we always assume that $p > 2 - \frac{1}{n}$, $\sigma \in (0,1)$, $\Omega \subset \mathbb{R}^n$ be an open, bounded and smooth domain. Furthermore, $u \in W^{1,\max\{1,p-1\}}(\Omega)$ is a SOLA solution to (1.1). Denote by $D(\Omega)$ the diameter of Ω, this means $D(\Omega) = \sup_{x,y\in\Omega} d(x,y)$.

First, suppose that $0 < R_0 < D(\Omega)/2$, let

$$\Omega_0 := \left\{ x \in \Omega \,\middle|\, 0 < d(x) \leq \frac{R_0}{2} \right\},$$

be the set of points near $\partial\Omega$. We define Ω_k as

$$\Omega_k := \left\{ x \in \Omega \,\middle|\, r_{k+1} < d(x) \leq r_k \right\},$$

with $r_k = 2^{-k}R_0, \forall k \in \mathbb{N}^\star$. It is clear that

$$\Omega_0 = \bigcup_{k=1}^{\infty} \Omega_k \quad \text{(see Figure 1).}$$

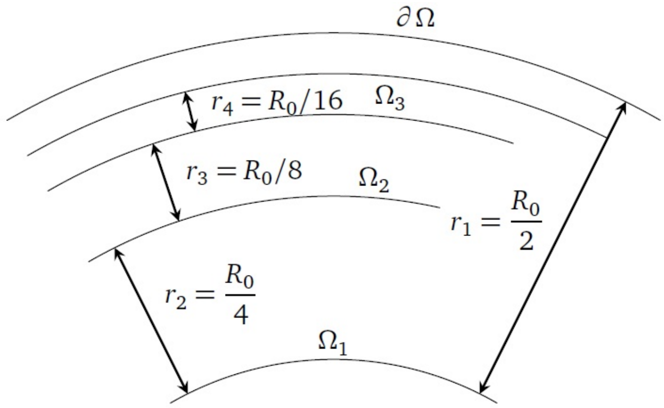

Figure 1. *The sets of points near the boundary*

To shorten notation, we introduce the following function

$$\mathbb{T}(x,y) := d^{\alpha}(x) d^{\beta}(y) \frac{|A(\nabla u(x)) - A(\nabla u(y))|}{|x-y|^{n+\sigma}}, \qquad x, y \in \Omega, \qquad x \neq y.$$

Let us introduce some lemmas that are necessary for later use.

Lemma 3.2. *Assume that* α, $\beta > 0$, $\gamma \geq 0$*;* $\alpha > \gamma$, $\beta > \gamma$ *and* $\alpha + \beta - \gamma > \sigma$*. Then, there exists a constant* $C > 0$ *such that*

$$\int_{\Omega \setminus \Omega_0} \int_{\Omega \setminus \Omega_0} \mathbb{T}(x,y) dx dy \leq C \left(\int_{\Omega \setminus \Omega_0} d^{\gamma}(x) \left| \mathcal{A}\left(\nabla u\left(x \right) \right) \right| dx + |\mu|\left(\Omega \setminus \Omega_0 \right) \right). \tag{3.2}$$

Proof of Lemma 3.2. First, let us establish

$$(\mathbb{III}) := \int_{\Omega \setminus \Omega_0} \int_{\Omega \setminus \Omega_0} \mathbb{T}(x,y) \mathrm{d}x \mathrm{d}y.$$

We remark that $\Omega \setminus \Omega_0$ can be covered by actually finite balls centered at z_k with radius r_1 , $k = \overline{1, N}$, i.e.

$$\Omega \setminus \Omega_0 \subset \bigcup_{k=1}^{N} B_{r_1}(z_k) = \bigcup_{z_k \in \Omega \setminus \Omega_0} B_{r_1}(z_k).$$

Let P be the set of all centers, i.e.

$$P := \{ z_k \in \Omega \setminus \Omega_0 : k \in \{1, 2, \ldots, N\} \}.$$

Now, we estimate $(\mathbb{III})$ as follows

$$(\mathbb{III}) = \int_{\Omega \setminus \Omega_0} \int_{\Omega \setminus \Omega_0} \mathbb{T}(x,y) \mathrm{d}x \mathrm{d}y \leq \sum_{z_k, z_l \in P} \int_{B_{r_1}(z_k)} \int_{B_{r_1}(z_l)} \mathbb{T}(x,y) \mathrm{d}x \mathrm{d}y.$$

Let P_{z_k} be the set of all centers that are closed to z_k, which means

$$P_{z_k} := \left\{ z_l \in P : B_{3r_1/2}(z_l) \cap B_{3r_1/2}(z_k) \neq \emptyset \right\}.$$

It is clear that

$$B_{r_1}(z_l) \subset B_{3r_1/2}(z_l) \subset B_{4r_1}(z_k), \quad \forall z_l \in P_{z_k} \text{ (see Figure 2).}$$

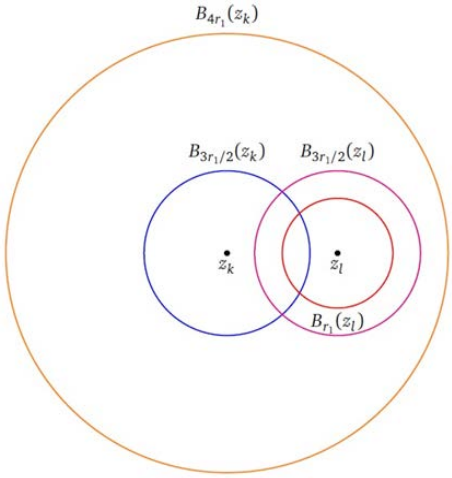

Figure 2. *The centers are closed to* z_k.

Furthermore, the cardinality of P_{z_k} is finite, i.e. there exists $C>0$ such that $|P_{z_k}| \le C$. So, we can decompose the integral $\Omega \setminus \Omega_0 \times \Omega \setminus \Omega_0$ as follows

$$\int_{\Omega\setminus\Omega_0}\int_{\Omega\setminus\Omega_0} \mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y \le \sum_{z_k, z_l \in P} \int_{B_{r_1}(z_k)}\int_{B_{r_1}(z_l)} \mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y$$

$$\le \sum_{z_k\in P}\sum_{z_l\in P_{z_k}} \int_{B_{r_1}(z_k)}\int_{B_{r_1}(z_l)} \mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y + \sum_{z_k\in P}\sum_{z_l\in P\setminus P_{z_k}} \int_{B_{r_1}(z_k)}\int_{B_{r_1}(z_l)} \mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y. \tag{3.3}$$

With the first term on the right-hand side of (3.3), we get

$$\sum_{z_k\in P}\sum_{z_l\in P_{z_k}} \int_{B_{r_1}(z_k)}\int_{B_{r_1}(z_l)} \mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y \le C\sum_{z_k\in P} \int_{B_{4r_1}(z_k)}\int_{B_{4r_1}(z_k)} \mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y. \tag{3.4}$$

Applying (1.5) in Theorem 1.1, we have

$$\int_{B_{4r_1}(z_k)}\int_{B_{4r_1}(z_k)} \mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y$$

$$\le 4^{\alpha+\beta-\gamma}\cdot r_1^{\alpha+\beta-\gamma}\int_{B_{4r_1}(z_k)}\int_{B_{4r_1}(z_k)} d^{\gamma}(x)\frac{|A(\nabla u(x))-A(\nabla u(y))|}{|x-y|^{n+\sigma}}\mathrm{d}x\mathrm{d}y$$

$$\le C.r_1^{\alpha+\beta-\gamma-\sigma}\left(\int_{B_{8r_1}(z_k)} d^{\gamma}(x)|A(\nabla u(x))|\mathrm{d}x + r_1[|\mu|(B_{8r_1})]\right). \tag{3.5}$$

Combining between (3.4) and (3.5), we reach that

$$\sum_{z_k\in P}\sum_{z_l\in P_{z_k}} \int_{B_{r_1}(z_k)}\int_{B_{r_1}(z_l)} \mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y$$

$$\leq C.r_1^{\alpha+\beta-\gamma-\sigma}\left(\sum_{z_k\in P}\int_{B_{8r_1}(z_k)} d^\gamma(x)\left|A(\nabla u(x))\right|\mathrm{d}x + r_1\sum_{z_k\in P}|\mu|(B_{8r_1})\right). \tag{3.6}$$

Notice that there is a constant $C = C(n) > 0$ such that

$$\sum_{z_k\in P}\chi_{B_{8r_1}(z_k)}(\xi) \leq C\chi_{\Omega\setminus\Omega_0}(\xi), \qquad \forall \xi\in\Omega,$$

therefore, for all $f\in L^1_{loc}(\mathbb{R}^n)$, we reach that

$$\sum_{z_k\in P}\int_{B_{8r_1}(z_k)} f(\xi)\mathrm{d}\xi = \sum_{z_k\in P}\int_{\mathbb{R}^n}\chi_{B_{8r_1}(z_k)}(\xi)f(\xi)\mathrm{d}\xi \leq C\int_{\Omega\setminus\Omega_0} f(\xi)\mathrm{d}\xi. \tag{3.7}$$

Substituting (3.7) to (3.6), we obtain that

$$\sum_{z_k\in P}\sum_{z_l\in P_{z_k}}\int_{B_{r_1}(z_k)}\int_{B_{r_1}(z_l)}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y$$
$$\leq C.r_1^{\alpha+\beta-\gamma-\sigma}\left(\int_{\Omega\setminus\Omega_0} d^\gamma(x)\left|A(\nabla u(x))\right|\mathrm{d}x + r_1[|\mu|(\Omega\setminus\Omega_0)]\right). \tag{3.8}$$

Moreover, it's clear that for any $x\in B_{r_1}(z_k)$, $y\in B_{r_1}(z_l)$, with $z_k\in P$ and $z_l\in P\setminus P_{z_k}$, we get $|x-y|\geq r_1$. It is easy for us to check that

$$\sum_{z_l\in P\setminus P_{z_k}}\int_{B_{r_1}(z_k)}\int_{B_{r_1}(z_l)} d^\alpha(x)d^\beta(y)\frac{|A(\nabla u(x))|}{|x-y|^{n+\sigma}}\mathrm{d}x\mathrm{d}y$$
$$\leq r_1^{\alpha+\beta-\gamma}\int_{B_{r_1}(z_k)}\left(\sum_{z_l\in P\setminus P_{z_k}}\int_{B_{r_1}(z_l)}\frac{1}{|x-y|^{n+\sigma}}\mathrm{d}y\right)d^\gamma(x)\left|A(\nabla u(x))\right|\mathrm{d}x$$
$$\leq r_1^{\alpha+\beta-\gamma-\sigma}\int_{B_{r_1}(z_k)}\left(\int_{\{|\xi|\geq 1\}}\frac{1}{|\xi|^{n+\sigma}}\mathrm{d}\xi\right)d^\gamma(x)\left|A(\nabla u(x))\right|\mathrm{d}x$$
$$\leq C.r_1^{\alpha+\beta-\gamma-\sigma}\int_{B_{r_1}(z_k)} d^\gamma(x)\left|A(\nabla u(x))\right|\mathrm{d}x.$$

Now we estimate the last term in (3.3) as

$$\sum_{z_k\in P}\sum_{z_l\in P\setminus P_{z_k}}\int_{B_{r_1}(z_k)}\int_{B_{r_1}(z_l)}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y \leq \mathrm{C}.r_1^{\alpha+\beta-\gamma-\sigma}\sum_{z_k\in P}\int_{B_{r_1}(z_k)} d^\gamma(x)\left|A(\nabla u(x))\right|\mathrm{d}x$$
$$\leq \mathrm{C}.r_1^{\alpha+\beta-\gamma-\sigma}\int_{\Omega\setminus\Omega_0} d^\gamma(x)\left|A(\nabla u(x))\right|\mathrm{d}x. \tag{3.9}$$

Applying (3.8), (3.9) to (3.3), we reach that

$$(\mathbb{III}) = \int_{\Omega\setminus\Omega_0}\int_{\Omega\setminus\Omega_0}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y$$
$$\leq C.\left(r_1^{\alpha+\beta-\gamma-\sigma}\int_{\Omega\setminus\Omega_0} d^\gamma(x)\left|A(\nabla u(x))\right|\mathrm{d}x + r_1^{\alpha+\beta-\gamma-\sigma+1}|\mu|(\Omega\setminus\Omega_0)\right)$$
$$\leq C.\left(\int_{\Omega\setminus\Omega_0} d^\gamma(x)\left|A(\nabla u(x))\right|\mathrm{d}x + |\mu|(\Omega\setminus\Omega_0)\right)r_1^{\alpha+\beta-\gamma-\sigma}$$

$$\leq C.\left(\int_{\Omega\setminus\Omega_0} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x+\left|\mu\right|(\Omega\setminus\Omega_0)\right), \tag{3.10}$$

which leads to the desired result. □

Lemma 3.3. *Assume that* α, $\beta>0$, $\gamma\geq 0$; $\alpha>\gamma$, $\beta>\gamma$ *and* $\alpha+\beta-\gamma>\sigma$. *Then, there exists a constant* $C>0$ *such that*

$$\int_{\Omega_i}\int_{\Omega_j} d^{\alpha}(x)d^{\beta}(y)\frac{\left|A(\nabla u(x))\right|}{\left|x-y\right|^{n+\sigma}}\mathrm{d}x\mathrm{d}y\leq C\frac{r_i^{\alpha-\gamma}r_j^{\beta}}{(r_i+r_j)^{\sigma}}\int_{\Omega_i} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x. \tag{3.11}$$

Proof of Lemma 3.3. First, for any $x\in\Omega_i$, $y\in\Omega_j$, $\left|i-j\right|\geq 2$, we get

$$\left|x-y\right|\geq\max\left\{\frac{r_i}{4},\frac{r_j}{4}\right\}\geq\frac{r_i+r_j}{8}\quad\text{(see Figure 3).}$$

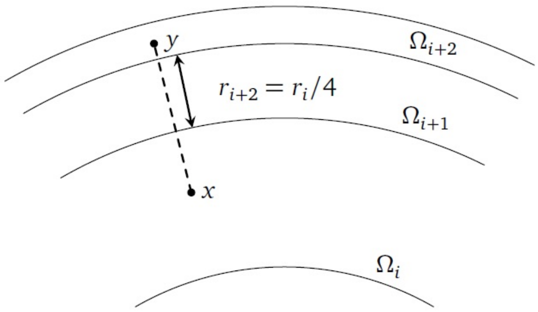

Figure 3. *Distance between* $x\in\Omega_i$ *and* $y\in\Omega_j$.

It yields

$$\int_{\Omega_i}\int_{\Omega_j} d^{\alpha}(x)d^{\beta}(y)\frac{\left|A(\nabla u(x))\right|}{\left|x-y\right|^{n+\sigma}}\mathrm{d}x\mathrm{d}y=\int_{\Omega_i}\int_{\Omega_j} d^{\alpha-\gamma}(x)d^{\beta}(y)\frac{\left|A(\nabla u(x))\right|}{\left|x-y\right|^{n+\sigma}}d^{\gamma}(x)\mathrm{d}x\mathrm{d}y$$

$$\leq r_i^{\alpha-\gamma}r_j^{\beta}\int_{\Omega_i}\left(\int_{\left\{\left|\xi\right|\geq\frac{r_i+r_j}{8}\right\}}\frac{1}{\left|\xi\right|^{n+\sigma}}\mathrm{d}\xi\right)d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x$$

$$\leq 8^{\sigma}\frac{r_i^{\alpha-\gamma}r_j^{\beta}}{(r_i+r_j)^{\sigma}}\int_{\Omega_i}\left(\int_{\left\{\left|\xi\right|\geq 1\right\}}\frac{1}{\left|\xi\right|^{n+\sigma}}\mathrm{d}\xi\right)d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x$$

$$\leq C\frac{r_i^{\alpha-\gamma}r_j^{\beta}}{(r_i+r_j)^{\sigma}}\int_{\Omega_i} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x. \tag{3.12}$$

Notice that the fraction $\frac{1}{\left|\xi\right|^{n+\sigma}}$ is integrable since $n+\sigma>n$. □

Lemma 3.4. *Assume that* α, $\beta>0$, $\gamma\geq 0$; $\alpha>\gamma$, $\beta>\gamma$ *and* $\alpha+\beta-\gamma>\sigma$. *Then, there exist constants* $C_1>0$ *and* $C_2>0$ *such that*

$$\sum_{i-j\geq 2}\left(\frac{r_i^{\alpha-\gamma}r_j^{\beta}}{(r_i+r_j)^{\sigma}}\int_{\Omega_i} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x+\frac{r_i^{\alpha}r_j^{\beta-\gamma}}{(r_i+r_j)^{\sigma}}\int_{\Omega_j} d^{\gamma}(y)\left|A(\nabla u(y))\right|\mathrm{d}y\right)$$

$$\le C_1.\int_{\Omega_0} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x\sum_{j=1}^{\infty} r_j^{\alpha+\beta-\gamma-\sigma}, \qquad (3.13)$$

and

$$\sum_{j-i\ge 2}\left(\frac{r_i^{\alpha-\gamma}r_j^{\beta}}{(r_i+r_j)^{\sigma}}\int_{\Omega_i} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x+\frac{r_i^{\alpha}r_j^{\beta-\gamma}}{(r_i+r_j)^{\sigma}}\int_{\Omega_j} d^{\gamma}(y)\left|A(\nabla u(y))\right|\mathrm{d}y\right)$$

$$\le C_2.\int_{\Omega_0} d^{\gamma}(y)\left|A(\nabla u(y))\right|\mathrm{d}y\sum_{i=1}^{\infty} r_i^{\alpha+\beta-\gamma-\sigma}. \qquad (3.14)$$

Proof of Lemma 3.4. First, let us establish

$$(\mathbb{I})_{11} := \sum_{i-j\ge 2}\left(\frac{r_i^{\alpha-\gamma}r_j^{\beta}}{(r_i+r_j)^{\sigma}}\int_{\Omega_i} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x+\frac{r_i^{\alpha}r_j^{\beta-\gamma}}{(r_i+r_j)^{\sigma}}\int_{\Omega_j} d^{\gamma}(y)\left|A(\nabla u(y))\right|\mathrm{d}y\right),$$

and

$$(\mathbb{I})_{12} := \sum_{j-i\ge 2}\left(\frac{r_i^{\alpha-\gamma}r_j^{\beta}}{(r_i+r_j)^{\sigma}}\int_{\Omega_i} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x+\frac{r_i^{\alpha}r_j^{\beta-\gamma}}{(r_i+r_j)^{\sigma}}\int_{\Omega_j} d^{\gamma}(y)\left|A(\nabla u(y))\right|\mathrm{d}y\right).$$

We have

$$(\mathbb{I})_{11} = \sum_{j=1}^{\infty}\sum_{i=j+2}^{\infty}\frac{r_i^{\alpha-\gamma}r_j^{\beta}}{\left(\frac{r_i}{r_j}+1\right)^{\sigma} r_j^{\sigma}}\int_{\Omega_i} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x$$

$$+\sum_{j=1}^{\infty}\sum_{i=j+2}^{\infty}\frac{r_i^{\alpha}r_j^{\beta-\gamma}}{\left(\frac{r_i}{r_j}+1\right)^{\sigma} r_j^{\sigma}}\int_{\Omega_j} d^{\gamma}(y)\left|A(\nabla u(y))\right|\mathrm{d}y$$

$$=\sum_{j=1}^{\infty} r_j^{\beta-\sigma}\sum_{i=j+2}^{\infty}\frac{r_i^{\alpha-\gamma}}{(2^{j-i}+1)^{\sigma}}\int_{\Omega_i} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x$$

$$+\sum_{j=1}^{\infty} r_j^{\beta-\gamma-\sigma}\int_{\Omega_j} d^{\gamma}(y)\left|A(\nabla u(y))\right|\mathrm{d}y\sum_{i=j+2}^{\infty}\frac{r_i^{\alpha}}{(2^{j-i}+1)^{\sigma}}$$

$$\le\sum_{j=1}^{\infty} r_j^{\alpha+\beta-\gamma-\sigma}\sum_{i=j+2}^{\infty}\int_{\Omega_i} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x+\sum_{j=1}^{\infty} r_j^{\beta-\gamma-\sigma}\int_{\Omega_j} d^{\gamma}(y)\left|A(\nabla u(y))\right|\mathrm{d}y\sum_{i=j+2}^{\infty} r_i^{\alpha}$$

$$\le\int_{\Omega_0} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x\sum_{j=1}^{\infty} r_j^{\alpha+\beta-\gamma-\sigma}+C_1.\sum_{j=1}^{\infty} r_j^{\alpha+\beta-\gamma-\sigma}\int_{\Omega_j} d^{\gamma}(y)\left|A(\nabla u(y))\right|\mathrm{d}y$$

$$\le C_1.\int_{\Omega_0} d^{\gamma}(x)\left|A(\nabla u(x))\right| \mathrm{d}x \sum_{j=1}^{\infty} r_j^{\alpha+\beta-\gamma-\sigma},$$

and similarly, we get

$$(\mathbb{I})_{12} = \sum_{j-i\ge 2}\left(\frac{r_i^{\alpha-\gamma} r_j^{\beta}}{(r_i+r_j)^{\sigma}}\int_{\Omega_i} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x + \frac{r_i^{\alpha} r_j^{\beta-\gamma}}{(r_i+r_j)^{\sigma}}\int_{\Omega_j} d^{\gamma}(y)\left|A(\nabla u(y))\right|\mathrm{d}y\right)$$

$$\le C_2.\int_{\Omega_0} d^{\gamma}(y)\left|A(\nabla u(y))\right| \mathrm{d}y \sum_{i=1}^{\infty} r_i^{\alpha+\beta-\gamma-\sigma},$$

which provides us (3.13) and (3.14). □

Lemma 3.5. *Assume that* α *,* $\beta > 0$*,* $\gamma \ge 0$*;* $\alpha > \gamma$*,* $\beta > \gamma$ *and* $\alpha+\beta-\gamma > \sigma$*. Then, there exists constant* $C > 0$ *such that*

$$\sum_{|i-j|\ge 2}\int_{\Omega_i}\int_{\Omega_j} \mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y \le C\int_{\Omega_0} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x\sum_{j=1}^{\infty} r_j^{\alpha+\beta-\gamma-\sigma}. \quad (3.15)$$

Proof of Lemma 3.5. In this proof, let us set

$$(\mathbb{I})_1 = \sum_{|i-j|\ge 2}\int_{\Omega_i}\int_{\Omega_j} \mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y.$$

Applying (3.11) in Lemma 3.3, we get

$$(\mathbb{I})_1 = \sum_{|i-j|\ge 2}\int_{\Omega_i}\int_{\Omega_j} \mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y$$

$$\le \sum_{|i-j|\ge 2}\left(\int_{\Omega_i}\int_{\Omega_j} d^{\alpha}(x)d^{\beta}(y)\frac{\left|A(\nabla u(x))\right|}{|x-y|^{n+\sigma}}\mathrm{d}x\mathrm{d}y + \int_{\Omega_i}\int_{\Omega_j} d^{\alpha}(x)d^{\beta}(y)\frac{\left|A(\nabla u(y))\right|}{|x-y|^{n+\sigma}}\mathrm{d}x\mathrm{d}y\right)$$

$$\le \sum_{|i-j|\ge 2}\left(\frac{r_i^{\alpha-\gamma} r_j^{\beta}}{(r_i+r_j)^{\sigma}}\int_{\Omega_i} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x + \frac{r_i^{\alpha} r_j^{\beta-\gamma}}{(r_i+r_j)^{\sigma}}\int_{\Omega_j} d^{\gamma}(y)\left|A(\nabla u(y))\right|\mathrm{d}y\right)$$

$$\le C.((\mathbb{I})_{11} + (\mathbb{I})_{12}), \quad (3.16)$$

where

$$(\mathbb{I})_{11} := \sum_{i-j\ge 2}\left(\frac{r_i^{\alpha-\gamma} r_j^{\beta}}{(r_i+r_j)^{\sigma}}\int_{\Omega_i} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x + \frac{r_i^{\alpha} r_j^{\beta-\gamma}}{(r_i+r_j)^{\sigma}}\int_{\Omega_j} d^{\gamma}(y)\left|A(\nabla u(y))\right|\mathrm{d}y\right),$$

and

$$(\mathbb{I})_{12} := \sum_{j-i\ge 2}\left(\frac{r_i^{\alpha-\gamma} r_j^{\beta}}{(r_i+r_j)^{\sigma}}\int_{\Omega_i} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x + \frac{r_i^{\alpha} r_j^{\beta-\gamma}}{(r_i+r_j)^{\sigma}}\int_{\Omega_j} d^{\gamma}(y)\left|A(\nabla u(y))\right|\mathrm{d}y\right).$$

From what have already been proved in (3.13), (3.14) and (3.16), $(\mathbb{I})_1$ can be estimated as

$$(\mathbb{I})_1 \le C\int_{\Omega_0} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x\sum_{j=1}^{\infty} r_j^{\alpha+\beta-\gamma-\sigma},$$

which allows us to get (3.15). □

Lemma 3.6. *Assume that* α, $\beta > 0$, $\gamma \ge 0$; $\alpha > \gamma$, $\beta > \gamma$ *and* $\alpha + \beta - \gamma > \sigma$. *Then, there exists constant* $C > 0$ *such that*

$$\sum_{i=1}^{\infty}\int_{\Omega_i}\int_{\Omega_i}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y \le C\left(\int_{\Omega_0} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x + |\mu|(\Omega_0)\right)\sum_{i=1}^{\infty} r_i^{\alpha+\beta-\gamma-\sigma}. \qquad (3.17)$$

Proof of Lemma 3.6. In this proof, let us denote

$$(\mathbb{I})_2 = \sum_{i=1}^{\infty}\int_{\Omega_i}\int_{\Omega_i}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y.$$

To continue estimates $(\mathbb{I})_2$, our idea is to decompose the Ω_i into open balls with a radius r_i then applying the local inequality (1.5) in Theorem 1.1.

Notice that Ω_i can be covered with $N_i \sim \frac{|\partial\Omega|}{r_i}$ balls centered at $z_k^i \in \Omega_i$ with radius r_i, $k = \overline{1, N_i}$. It means

$$\Omega_i \subset \bigcup_{k=1}^{N_i} B_{r_i}(z_k^i) = \bigcup_{z_k^i \in \Omega_i} B_{r_i}(z_k^i).$$

Let P_i be the set of all centers, i.e.

$$P_i := \{z_k^i \in \Omega_i : k \in \{1, 2, \ldots, N_i\}\}.$$

Now, we estimate $(\mathbb{I})_2$ as follows

$$(\mathbb{I})_2 = \sum_{i=1}^{\infty}\int_{\Omega_i}\int_{\Omega_i}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y \le \sum_{i=1}^{\infty}\sum_{z_k^i, z_l^i \in P_i}\int_{B_{r_i}(z_k^i)}\int_{B_{r_i}(z_l^i)}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y.$$

Let P_{i,z_k^i} be the set of all centers that are closed to z_k^i, which means

$$P_{i,z_k^i} := \left\{z_l^i \in P_i : B_{3r_i/2}(z_l^i) \cap B_{3r_i/2}(z_k^i) \ne \varnothing\right\}.$$

It is not difficult for us to check that

$$B_{r_i}(z_l^i) \subset B_{3r_i/2}(z_l^i) \subset B_{4r_i}(z_k^i), \forall z_l^i \in P_{i,z_k^i}.$$

Moreover, the cardinality of P_{i,z_k^i} is finite, means there exists a constant C such that $\left|P_{i,z_k^i}\right| \le C$. So, we can decompose the integral $\Omega_i \times \Omega_i$ as follows

$$\int_{\Omega_i}\int_{\Omega_i}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y \le \sum_{z_k^i, z_l^i \in P_i}\int_{B_{r_i}(z_k^i)}\int_{B_{r_i}(z_l^i)}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y$$

$$\leq \sum_{z_k^i \in P_i} \sum_{z_l^i \in P_{i,z_k^i}} \int_{B_{r_i}(z_k^i)} \int_{B_{r_i}(z_l^i)} \mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y$$

$$+ \sum_{z_k^i \in P_i} \sum_{z_l^i \in P_i \setminus P_{i,z_k^i}} \int_{B_{r_i}(z_k^i)} \int_{B_{r_i}(z_l^i)} \mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y. \quad (3.18)$$

Applying (3.8), (3.9) to (3.18), we reach that

$$(\mathbb{I})_2 = \sum_{i=1}^{\infty} \int_{\Omega_i} \int_{\Omega_i} \mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y$$

$$\leq C.\left(\sum_{i=1}^{\infty} r_i^{\alpha+\beta-\gamma-\sigma} \int_{\Omega_0} d^{\gamma}(x) \left| A(\nabla u(x)) \right| \mathrm{d}x + r_i^{\alpha+\beta-\gamma-\sigma+1} |\mu|(\Omega_0) \right)$$

$$\leq C.\left(\int_{\Omega_0} d^{\gamma}(x) \left| A(\nabla u(x)) \right| \mathrm{d}x + |\mu|(\Omega_0) \right) \sum_{i=1}^{\infty} r_i^{\alpha+\beta-\gamma-\sigma}, \quad (3.19)$$

which leads to the desired result. □

Lemma 3.7. *Assume that* α *,* $\beta > 0$ *,* $\gamma \geq 0$ *;* $\alpha > \gamma$ *,* $\beta > \gamma$ *and* $\alpha + \beta - \gamma > \sigma$ *. Then, there exists constant* $C > 0$ *such that*

$$\sum_{|i-j|=1} \int_{\Omega_i} \int_{\Omega_j} \mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y \leq C\left(\int_{\Omega_0} d^{\gamma}(x) \left| A(\nabla u(x)) \right| \mathrm{d}x + \mu(\Omega_0) \right) \sum_{i=1}^{\infty} r_i^{\alpha+\beta-\gamma-\sigma}. \quad (3.20)$$

Proof of Lemma 3.7. Let us establish

$$(\mathbb{I})_3 := \sum_{|i-j|=1} \int_{\Omega_i} \int_{\Omega_j} \mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y.$$

We estimate $(\mathbb{I})_3$ with a note that

$$(\mathbb{I})_3 = \sum_{|i-j|=1} \int_{\Omega_i} \int_{\Omega_j} \mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y = 2\sum_{i=1}^{\infty} \int_{\Omega_i} \int_{\Omega_{i+1}} \mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y \leq 2\sum_{i=1}^{\infty} \int_{A_i} \int_{A_i} \mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y,$$

where A_i is defined by

$$A_i := \Omega_i \cup \Omega_{i+1} = \left\{ x \in \Omega : \frac{r_i}{4} < d(x) \leq r_i \right\}.$$

Similarly, for $(\mathbb{I})_3$ we may estimate by the same the way to $(\mathbb{I})_2$ in (3.18) and reach that

$$(\mathbb{I})_3 \leq C.\left(\int_{\Omega_0} d^{\gamma}(x) \left| A(\nabla u(x)) \right| \mathrm{d}x + \mu(\Omega_0) \right) \sum_{i=1}^{\infty} r_i^{\alpha+\beta-\gamma-\sigma}, \quad (3.21)$$

which leads to the desired result. □

Lemma 3.8. *Assume that* α *,* $\beta > 0$ *,* $\gamma \geq 0$ *;* $\alpha > \gamma$ *,* $\beta > \gamma$ *and* $\alpha + \beta - \gamma > \sigma$ *. Then, there exists a constant* $C > 0$ *such that*

$$\int_{\Omega_0}\int_{\Omega_0}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y \le C\left(\int_{\Omega_0} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x + |\mu|(\Omega_0)\right). \tag{3.22}$$

Proof of Lemma 3.8. In this proof, let us set

$$(\mathbb{I}) := \int_{\Omega_0}\int_{\Omega_0}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y.$$

Since $\Omega_0 = \bigcup_{k=1}^{\infty}\Omega_k$, we can rewrite $(\mathbb{I})$ as follows

$$\begin{aligned}(\mathbb{I}) = \sum_{i,j=1}^{\infty}\int_{\Omega_i}\int_{\Omega_j}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y &= \sum_{|i-j|\ge 2}\int_{\Omega_i}\int_{\Omega_j}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y + \sum_{|i-j|=1}\int_{\Omega_i}\int_{\Omega_j}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y \\ &\quad + \sum_{i=1}^{\infty}\int_{\Omega_i}\int_{\Omega_i}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y \\ &= (\mathbb{I})_1 + (\mathbb{I})_3 + (\mathbb{I})_2,\end{aligned} \tag{3.23}$$

with

$$(\mathbb{I})_1 = \sum_{|i-j|\ge 2}\int_{\Omega_i}\int_{\Omega_j}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y; \quad (\mathbb{I})_3 = \sum_{|i-j|=1}\int_{\Omega_i}\int_{\Omega_j}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y$$

and

$$(\mathbb{I})_2 = \sum_{i=1}^{\infty}\int_{\Omega_i}\int_{\Omega_i}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y.$$

We can estimate each term on the right-hand side of (3.23) by applying Lemma 3.5, Lemma 3.6 and Lemma 3.7. Then, we can find a constant $C > 0$ such that

$$(\mathbb{I}) \le C\left(\int_{\Omega_0} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x + |\mu|(\Omega_0)\right). \tag{3.24}$$

Note that the assumption $\alpha+\beta-\gamma > \sigma$ helps us to find

$$\sum_{i=1}^{\infty} r_i^{\alpha+\beta-\gamma-\sigma} = CR_0^{\alpha+\beta-\gamma-\sigma}, \quad \text{with } C = \sum_{i=1}^{\infty}\left(\frac{1}{2}\right)^{(\alpha+\beta-\gamma-\sigma)i} < \infty,$$

which completes the proof. □

Lemma 3.9. *For every α, $\beta > 0$, $\gamma \ge 0$ satisfying $\alpha > \gamma$, $\beta > \gamma$ and $\alpha+\beta-\gamma > \sigma$, there exists a constant $C > 0$ such that*

$$\int_{\Omega_0}\int_{\Omega\setminus\Omega_0}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y \le C\left(\int_{\Omega\setminus\Omega_0} d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x + |\mu|(\Omega\setminus\Omega_0)\right). \tag{3.25}$$

Proof of Lemma 3.9. Note that Ω_i can be covered with $N_i \sim \frac{|\partial\Omega|}{r_i}$ balls centered at z_l^i with radius r_i, i.e.

$$\Omega_i \subset \bigcup_{l=1}^{N_i} B_{r_i}(z_l^i) = \bigcup_{z_l^i \in P_i} B_{r_i}(z_l^i),$$

and $\Omega\setminus\Omega_0$ can be covered by finite balls centered at z_k with radius r_1, i.e.

$$\Omega\setminus\Omega_0 \subset \bigcup_{k=1}^{N} B_{r_1}(z_k) = \bigcup_{z_k\in P} B_{r_1}(z_k),$$

where

$$P_i := \{z_l^i \in \Omega_i : l\in\{1,2,\dots,N_i\}\} \quad \text{and} \quad P := \{z_k\in\Omega\setminus\Omega_0 : k\in\{1,2,\dots,N\}\}.$$

It is not difficult for us to check that

$$B_{r_i}(z_l^i) \subset B_{4r_1}(z_k), \forall z_l^i \in P_i.$$

Now, we estimate $(\mathbb{II})$ as follows

$$\begin{aligned}(\mathbb{II}) &= \int_{\Omega_0}\int_{\Omega\setminus\Omega_0}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y = \sum_{i=1}^{\infty}\int_{\Omega_i}\int_{\Omega\setminus\Omega_0}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y\\ &\le \sum_{i=1}^{\infty}\sum_{z_l^i\in P_i}\sum_{z_k\in P}\int_{B_{r_i}(z_l^i)}\int_{B_{r_1}(z_k)}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y = \sum_{i=1}^{\infty}\sum_{z_k\in P}\sum_{z_l^i\in P_i}\int_{B_{r_1}(z_k)}\int_{B_{r_i}(z_l^i)}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y\\ &\le C\sum_{z_k\in P}\int_{B_{4r_1}(z_k)}\int_{B_{4r_1}(z_k)}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y. \end{aligned} \tag{3.26}$$

Combining between (3.5) and (3.26), we reach that

$$(\mathbb{II}) \le C(n,p,c_A,\sigma,R_0)r_1^{\alpha+\beta-\gamma-\sigma}\left(\sum_{z_k\in P}\int_{B_{8r_1}(z_k)}d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x + r_1\sum_{z_k\in P}|\mu|(B_{8r_1})\right). \tag{3.27}$$

Substituting (3.7) to (3.27), we obtain that

$$\begin{aligned}(\mathbb{II}) &\le C(n,p,c_A,\sigma,R_0)r_1^{\alpha+\beta-\gamma-\sigma}\left(\int_{\Omega\setminus\Omega_0}d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x + r_1[|\mu|(\Omega\setminus\Omega_0)]\right)\\ &\le C(n,p,c_A,\alpha,\beta,\gamma,\sigma,R_0)\left(\int_{\Omega\setminus\Omega_0}d^{\gamma}(x)\left|A(\nabla u(x))\right|\mathrm{d}x + |\mu|(\Omega\setminus\Omega_0)\right). \end{aligned} \tag{3.28}$$

This achieves the proof of the desired result. □

Thanks to some lemmas that have been proved and some important properties of weighted fractional Sobolev spaces discussed in Section 2, now we prove the main theorem.

Proof of Theorem 3.1. The integral of $\mathbb{T}$ over $\Omega\times\Omega$ can be rewritten as

$$\begin{aligned}\int_{\Omega}\int_{\Omega}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y &= \int_{\Omega_0}\int_{\Omega_0}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y + 2\int_{\Omega_0}\int_{\Omega\setminus\Omega_0}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y + \int_{\Omega\setminus\Omega_0}\int_{\Omega\setminus\Omega_0}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y\\ &= (\mathbb{I}) + 2(\mathbb{II}) + (\mathbb{III}), \end{aligned} \tag{3.29}$$

with

$$(\mathbb{I}) = \int_{\Omega_0}\int_{\Omega_0}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y; \qquad (\mathbb{II}) = \int_{\Omega_0}\int_{\Omega\setminus\Omega_0}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y,$$

and

$$(\mathbb{III}) = \int_{\Omega\setminus\Omega_0}\int_{\Omega\setminus\Omega_0}\mathbb{T}(x,y)\mathrm{d}x\mathrm{d}y.$$

We can estimate each term $(\mathbb{I})$, $(\mathbb{II})$ and $(\mathbb{III})$ by using Lemma 3.2, Lemma 3.8, and Lemma 3.9. Then, there exists constant $C = C(n, p, c_{\mathcal{A}}, \alpha, \beta, \gamma, \sigma, R_0) > 0$ such that

$$\int_\Omega \int_\Omega \mathbb{T}(x, y)\mathrm{d}x\mathrm{d}y \leq \mathrm{C}\left(\int_\Omega d^\gamma(x)\left|\mathcal{A}(\nabla u(x))\right|\mathrm{d}x + |\mu|(\Omega)\right), \tag{3.30}$$

which leads to the desired result (3.1) from (3.30). ☐

❖ ***Conflict of Interest:*** *Author have no conflict of interest to declare.*

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BẤT ĐẲNG THỨC CACCIOPOLI CÓ TRỌNG CHO NGHIỆM CỦA PHƯƠNG TRÌNH P-LAPLACE

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TÓM TẮT

Không gian Sobolev cấp phân số có trọng có nhiều ứng dụng trong phương trình đạo hàm riêng. Trong bài báo này, chúng tôi khảo sát lớp không gian Sobolev cấp phân số có trọng, ứng với hàm trọng là hàm khoảng cách đến biên của miền xác định. Lớp không gian này được sử dụng để thu được một dạng bất đẳng thức dạng Cacciopoli có trọng cho bài toán p-Laplace với dữ liệu độ đo. Kết quả của chúng tôi là mở rộng của bất đẳng thức Cacciopoli trong bài báo gần đây (Tran & Nguyen, 2021b).

Từ khóa: bất đẳng thức dạng Cacciopoli; phương trình đạo hàm riêng; phương trình *p*-Laplace; không gian Sobolev cấp phân số có trọng