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Research Article

TRIEBEL-LIZORKIN-MORREY SPACES ASSOCIATED WITH NONNEGATIVE SELF-ADJOINT OPERATOR

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ABSTRACT

Let L be a nonnegative self-adjoint operator on $L^2(\mathbb{R}^n)$ *with a heat kernel satisfying a Gaussian upper bound. In this work, we introduce Triebel-Lizorkin-Morrey spaces* $FM^{\alpha,L}_{p,q}(\mathbb{R}^n)$ *associated with the operator L for the entire range* $0 < p,q \le \infty, \alpha \in \mathbb{R}$. *We then prove that our new spaces satisfy important features such as continuous characterizations in terms of square functions, or atomic decomposition.*

Keywords: atomic decompositions; continuous characterization; Gaussian upper bound; Triebel-Lizorkin-Morrey space

1. Introduction

The classical Triebel-Lizorkin spaces on Euclidean spaces $\mathbb{R}^n$, considered as generalizations of other classical spaces such as Lesbegue spaces, BMO spaces, Hardy spaces, and Sobolev spaces, are essential in approximation theory and partial differential equations. Recently, the theory of new Triebel-Lizorkin spaces associated with differential operators *L* has been developed by many mathematicians in various settings. We summarize here some remarkable literature related to this new research development:

- Using the existence of the approximation of identity, Han and Sawyer (1994) developed a theory of Triebel-Lizorkin spaces $F^s_{p,q}$ for a range $1 \le p,q \le \infty$ and $s \in (-\theta, \theta)$ for some $\theta \in (0,1)$ on metric measure spaces with polynomial volume growths.
- Petrushev and Xu (2008) introduced new Triebel-Lizorkin spaces associated with the Hermite operator with a full range of indices. They then proved the frame decompositions for these spaces by making use of estimates of eigenvectors of the Hermite operators. Similar

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results were also proved for the Laguerre operator by Kerkyacharian et al.(2009). The theory of these function spaces was further developed by Bui and Duong (2017) in which the authors proved molecular and atomic decompositions theorems and square function characterizations for these spaces.

- Under the assumption that L is a nonnegative self-adjoint operator satisfying Gaussian upper bounds, Holder continuity, and Markov semigroup properties, the frame decompositions of Triebel-Lizorkin spaces associated with L with a full range of indices were studied in Georgiadis et al. (2017). This theory has a wide range of applications from the setting of Lie groups to Riemannian manifolds.
- Bui et al. (2020) established a theory of weighted Besov and Triebel-Lizorkin spaces associated with a nonnegative self-adjoint operator. In contrast to Georgiadis et al. (2017), the authors assumed the Gaussian upper bound, but did not assume Holder continuity on the heat kernels nor the Markov properties (the conservation property). This allows their theory to cover a wider range of applications including regularity estimates for certain singular integrals with rough kernels which are beyond the class of Calderon-Zygmund operators.

On the other hand, many authors have extended the theory of Triebel-Lizorkin spaces to the setting Triebel-Lizorkin-Morrey (TLM in abbreviation) by using Morrey spaces in place of $L^p(\mathbb{R}^n)$ in the definition of Triebel-Lizorkin spaces, as they realized that TLM spaces share key properties of Triebel-Lizorkin spaces, and represent local oscillations and singularities of functions better than Triebel-Lizorkin spaces do. We also list here some important results related to this research direction:

- Tang and Xu (2005) introduced the inhomogeneous TLM spaces and studied lifting properties, Fourier multiplier theorem, and the discrete characterization of inhomogeneous TLM spaces.
- Sawano (2008) characterized the inhomogeneous TLM spaces in terms of wavelet.
- Wang (2009) established the decomposition of homogeneous TLM in terms of molecules.
- Nguyen et al. (2020) investigated TLM spaces associated with Hermite operators by adapting the technique of maximal functions introduced by Fefferman-Stein and Peetre, whereas usual approaches for these types of function spaces are based on Littlewood-Paley decompositions. As a result of this distinct approach, it is possible to extend the theory of the inhomogeneous TLM spaces to the setting where p and q are below the endpoint 1.

Let L be a nonnegative, self-adjoint operator on $L^2(\mathbb{R}^n)$ which generates a semigroup $\left(e^{-tL}\right)_{t>0}$, and $p_t(x,y)$ be the kernel of this semigroup. Throughout the paper, we always assume that the kernel $p_t(x,y)$ holds a Gaussian upper bound (GUB), i.e., there exist two positive constants C and c such that for all $x, y \in \mathbb{R}^n$ and $t > 0$,

$$|p_t(x,y)| \le \frac{C}{t^{n/2}} \exp\Big(-\frac{|x-y|^2}{ct}\Big).$$

Noting that such an operator L covers the classical Schrodinger operators associated with nonnegative potentials satisfying the reverse Holder inequality on $\mathbb{R}^n$, our main aim in this paper is to develop the theory of new Triebel–Lizorkin-Morrey spaces associated with such a nonnegative, self-adjoint operator satisfying the Gaussian upper bound. It should be pointed out that in this work, we do not assume the additional conditions such as Holder continuity estimate and Markov semigroup property for the setting Lebesgue spaces. So as to be able to present the new TLM spaces, we use spectral decompositions of nonnegative self-adjoint operators. Moreover, the extension of relevant classical techniques and tools to the current setting is nontrivial, including new ideas concerning new space of distributions and several estimates related to maximal functions associated with functional calculus of L which were established recently. These are interests in their own right and should be useful in future research in the field.

Our paper is organized as follows: Section 2 recalls preliminaries, class of distributions, and related estimates. Section 3 presents the definition of new TLM spaces and the proof of continuous characterizations of new TLM spaces in terms of square functions via heat kernels. Section 4 establishes atomic decompositions results of these new spaces.

Throughout the paper, we use C and c to denote positive constants that are independent of the main parameters involved but whose values may differ from line to line. We write $A \lesssim B$ if there is a universal positive constant C so that $A \le CB$ and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. Set $\mathbb{N} = \{1;2;3;...\}$ and $\mathbb{N}_+ = \mathbb{N} \cup \{0\}$. For $1 \le p \le \infty$, denote by p' the conjugate exponent of p, i.e. $\frac{1}{p} + \frac{1}{p'} = 1$. In addition, given $\lambda > 0$ and a ball $B = B(x_B, r_B)$, we write λB for the λ-dilated ball, which is the ball with the same center as B and with the radius $r_{\lambda B} = \lambda r_B$. For each ball $B \subset \mathbb{R}^n$, we set $S_0(B) = 4B$ and $S_j(B) = 2^j B \setminus 2^{j-1} B$ for $j \ge 3$. Finally, for $a, b \in \mathbb{R}$, let $a \wedge b = \min\{a,b\}$ and $a \vee b = \max\{a,b\}$.

2. Preliminaries, class of distributions, and related estimates

2.1. Dyadic cubes

Firstly, let us recall the set of all dyadic cubes $\mathcal{D}$ in $\mathbb{R}^n$

$$\mathcal{D} = \left\{ \prod_{j=1}^{n} \big[m_j 2^k, (m_j+1)2^k \big) : m_1, m_2, ..., m_n, k \in \mathbb{Z} \right\}.$$

For a dyadic cube $Q := \prod_{j=1}^{n} \left[m_j 2^k, (m_j+1)2^k \right)$, we denote by $\ell(Q)$ and x_Q the length and the center of the dyadic cube Q respectively. For $v \in \mathbb{Z}$, we set $\mathcal{D}_v = \left\{ Q \in \mathcal{D} : \ell(Q) = 2^v \right\}$.

2.2. The Hardy-Littlewood maximal function

Let $0 < \theta < \infty$. The Hardy-Littlewood maximal function $\mathcal{M}_\theta$ be defined by

$$\mathcal{M}_\theta f(x) = \sup_{x \in B} \left(\frac{1}{|B|} \int_B |f(y)|^\theta \, dy \right)^{1/\theta},$$

where the supremum is taken over all balls B containing x. The subscript θ is dropped when $\theta = 1$.

The following elementary estimates will be used in the sequel (see, for example, Bui et al., 2018).

Lemma 2.1. *Let* $s, \varepsilon > 0$ *and* $p \in [1, \infty]$.

i. For all $x \in \mathbb{R}^n$, *we have:* $\left(\int_{\mathbb{R}^n} \left[\left(1 + |x-y|/s \right)^{-n-\epsilon} \right]^p dy \right)^{1/p} \lesssim s^{n/p}$.

ii. For any $f \in L^1_{loc}\left(\mathbb{R}^n\right), x \in \mathbb{R}^n$, *we have:* $\int_{\mathbb{R}^n} \frac{1}{s^n} \left(1 + |x-y|/s \right)^{-n-\epsilon} |f(y)| \, dy \lesssim \mathcal{M}f(x)$.

2.3. Morrey space

We recall here some important estimates involving Morrey spaces which are used in the following sections.

Lemma 2.2. [Trong et al., 2020, Lemma 2.5]

The following statements hold true:

i. For $0 < p \le r < \infty$, *we have:* $\| f \|_{\mathrm{M}^r_p} \sim \sup_{Q \in \mathcal{D}} |Q|^{1/r - 1/p} \| f \|_{L^p(Q)}$.

ii. For any $0 < p \le r < \infty, \theta > 0$, *we have:* $\| f^\theta \|_{\mathrm{M}^r_p} \lesssim \| f \|^\theta_{\mathrm{M}^{r\theta}_{p\theta}}$.

iii. (Minkowski's inequality) For any $0 < q \le p \le r < \infty$, *we have:*

$$\left\| \left(\int_a^b |F(\cdot,t)|^q \frac{dt}{t} \right)^{1/q} \right\|_{\mathrm{M}^r_p} \lesssim \left(\int_a^b \| F(\cdot,t) \|^q_{\mathrm{M}^r_p} \frac{dt}{t} \right)^{1/q}.$$

The next lemma (the Fefferman-Stein vector-valued maximal inequality) plays a key role in this paper.

Lemma 2.3. [Trong et al., 2020, Lemma 2.6] *Let* $0 < p \le r \le \infty$, $0 < q \le \infty$ *and* $0 < \theta < p \wedge q$. *Then for any sequence of measurable functions* $\{f_v\}_{v \in \mathbb{Z}}$, *we have*

$$\left\| \left(\sum_{v \in \mathbb{Z}} |\mathcal{M}_\theta(f_v)|^q \right)^{1/q} \right\|_{\mathrm{M}^r_p} \lesssim \left\| \left(\sum_{v \in \mathbb{Z}} |f_v|^q \right)^{1/q} \right\|_{\mathrm{M}^r_p}. \quad (2.1)$$

Remark 2.4. As a consequence of Lemma 2.3, if $0<p\le r\le\infty$ and $0<\theta<p\wedge 1$ then the Hardy-Littlewood maximal operator $\mathcal{M}_\theta$ is bounded on M^r_p. In addition, for $(a_\nu)\in\ell^q\cap\ell^1$, $0<p\le r\le\infty$, $0<q\le\infty$ and $0<\theta<p\wedge q$, one has

$$\Big\|\sum_j\Big(\sum_\nu|a_{j-\nu}\mathcal{M}_\theta(f_\nu)|^q\Big)^{1/q}\Big\|_{M^r_p}\lesssim\Big\|\Big(\sum_\nu|f_\nu|^q\Big)^{1/q}\Big\|_{M^r_p}. \tag{2.2}$$

2.4. Kernel estimates

Denote by $E_L(\lambda)$ a spectral decomposition of L. Then by spectral theory, for any bounded Borel funtion $F:[0,\infty)\to\mathbb{C}$, we can define

$$F(L)=\int_0^\infty F(\lambda)dE_L(\lambda)$$

as a bounded operator on $L^2(\mathbb{R}^n)$. It is well-known that $\operatorname{supp}K_{\cos(t\sqrt{L})}\subset\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n:|x-y|\le t\}$, where $K_{\cos(t\sqrt{L})}$ is the kernel of $\cos(t\sqrt{L})$.

We have the following useful lemma (see, for example, Hofmann et al., 2011).

Lemma 2.5. *Let $\varphi\in S(\mathbb{R})$ be an even function with $\operatorname{supp}\varphi\subset(-1,1)$ and $\int\varphi=2\pi$. Denote by Φ the Fourier transform of φ. Then for every $k\in\mathbb{N}$, the kernel $K_{(t^2L)^k\Phi(t\sqrt{L})}$ of $(t^2L)^k\Phi(t\sqrt{L})$ satisfies $\operatorname{supp}K_{(t^2L)^k\Phi(t\sqrt{L})}\subset\{(x,y)\in\mathbb{R}^n\times\mathbb{R}^n:|x-y|\le t\}$, and $|K_{(t^2L)^k\Phi(t\sqrt{L})}(x,y)|\le\dfrac{C}{t^n}$.*

The next lemma provides some key kernel estimates which play an important role in establishing our main results.

Lemma 2.6. [Bui et al., 2020, Lemma 2.6]

i. Let $\varphi\in S(\mathbb{R})$ be an even function. Then for any $N>0$, there exists $C>0$ such that for all $t>0$ and $x,y\in\mathbb{R}^n$, we have: $|K_{\varphi(t\sqrt{L})}(x,y)|\le Ct^{-n}\big(1+x-y|/t\big)^{-N}$.

ii. Let $\varphi_1,\varphi_2\in S(\mathbb{R})$ be even functions. Then for any $N>0$, there exists $C>0$ such that for all $t\le s<2t$ and $x,y\in\mathbb{R}^n$, we have: $|K_{\varphi_1(t\sqrt{L})\varphi_2(s\sqrt{L})}(x,y)|\le Ct^{-n}\big(1+|x-y|/t\big)^{-N}$.

iii. Let $\varphi_1,\varphi_2\in S(\mathbb{R})$ be even functions with $\varphi_2^{(\nu)}(0)=0$ for $\nu=0,1,\dots,2\ell$ for some $\ell\in\mathbb{Z}^+$. Then for any $N>0$, there exists $C>0$ such that for all $t\ge s>0$ and $x,y\in\mathbb{R}^n$, we have: $|K_{\varphi_1(t\sqrt{L})\varphi_2(s\sqrt{L})}(x,y)|\le C\big(s/t\big)^{2\ell}t^{-n}\big(1+|x-y|/t\big)^{-N}$.

Remark 2.7. Note that any function in $S(\mathbb{R})$ with compact support in $(0,\infty)$ can be extended to an even function in $S(\mathbb{R})$ with derivatives of all orders vanishing at 0. Hence, the results in Lemma 2.6 hold for such functions.

2.5. A new class of distributions

The class of test functions $\mathcal{S}$ associated with L is defined as the set of all functions $\phi \in \cap_{m\geq 1} D(L^m)$ such that $\mathcal{P}_{m,\ell}(\phi) = \sup_{x\in\mathbb{R}^n}(1+|x|)^m |L^\ell \phi(x)| < \infty,\ \forall m > 0, \ell \in \mathbb{N}.$

$\mathcal{S}$ is a complete locally convex space with topology generated by the family of semi-norms $\{\mathcal{P}_{m,\ell} : m > 0, \ell \in \mathbb{N}\}$ (see Keryacharian & Petrushev, 2015).

As usual, we define the space of distributions $\mathcal{S}'$ as the set of all continuous linear functionals on $\mathcal{S}$ with the inner product defined by $\langle f, \phi\rangle = f(\phi)$, for all $f \in \mathcal{S}'$ and $\phi \in \mathcal{S}$.

Define by $\mathcal{S}_\infty$ the space of all functions $\phi \in \mathcal{S}$ such that for each $k \in \mathbb{N}$ there exists $g_k \in \mathcal{S}$ so that $\phi = L^k g_k$. Note that such an g_k, if exists, is unique (see Georgiadis et al., 2017). The topology in $\mathcal{S}_\infty$ is generated by the following family of semi-norms $\mathcal{P}^*_{m,\ell,k}(\phi) = \mathcal{P}_{m,\ell}(g_k), m > 0, \ell, k \in \mathbb{N},$ where $\phi = L^k g_k$.

We then denote by $\mathcal{S}'_\infty$ the set of all continuous linear functionals on $\mathcal{S}_\infty$. In order to have an insightful understanding about the distributions in $\mathcal{S}'_\infty$, one sets $\mathbb{P}_m = \{g \in \mathcal{S}' : L^m g = 0\}, m \in \mathbb{N},$ and set $\mathbb{P} = \cup_{m\in\mathbb{N}} \mathbb{P}_m$.

Based onProposition 3.7 in Georgiadis et al. (2017) we have the following identification:

Proposition 2.8. $\mathcal{S}'/\mathbb{P} = \mathcal{S}'_\infty$.

It was also proved in Georgiadis et al., 2017 that with $L = -\Delta$, the Laplacian on $\mathbb{R}^n$, the distributions in $\mathcal{S}'/\mathbb{P} = \mathcal{S}'_\infty$ are identical with the classical tempered distributions modulo polynomial.

From Lemma 2.6, we can see that if $\varphi \in S(\mathbb{R})$ with $\operatorname{supp}\varphi \subset (0,\infty)$, then we have $K_{\varphi(t\sqrt{L})}(x,\cdot) \in \mathcal{S}_\infty$ and $K_{\varphi(t\sqrt{L})}(\cdot, y) \in \mathcal{S}_\infty$. Therefore, we can define for all $f \in \mathcal{S}'_\infty$

$$\varphi(t\sqrt{L}) f(x) = \langle f, K_{\varphi(t\sqrt{L})}(x,\cdot)\rangle. \qquad (2.3)$$

Remark 2.9. The support condition $\operatorname{supp}\varphi \subset (0,\infty)$ is essential to be able to define $\varphi(t\sqrt{L}) f$ with

$f \in \mathcal{S}'_\infty$. In general, if $\varphi \in S(\mathbb{R})$ then we have $K_{\varphi(t\sqrt{L})}(x,\cdot) \in \mathcal{S}$ and $K_{\varphi(t\sqrt{L})}(\cdot, y) \in \mathcal{S}$.

Lemma 2.10. [Bui et al., 2020, Lemma 2.9]

Let $f \in \mathcal{S}'$ and $\varphi \in S(\mathbb{R})$ *be an even function. Then there exist* $m > 0$ *and* $K > 0$ *such that*

$$|\varphi(t\sqrt{L}) f(x)| \lesssim \frac{(t\vee t^{-1})^m}{t^n}(1+|x|)^K. \qquad (2.4)$$

The similar estimate holds true if $f \in \mathcal{S}'_\infty$ and $\varphi \in S(\mathbb{R})$ *supported in* $[1/2, 2]$.

2.6. Maximal function estimates

For $\lambda > 0, j \in \mathbb{Z}$ and $\varphi \in S(\mathbb{R})$, the Peetre's type maximal function is defined by

$$\varphi^*_{j,\lambda}(\sqrt{L}) f(x) = \sup_{y\in\mathbb{R}^n} \frac{|\varphi_j(\sqrt{L}) f(y)|}{(1+2^j|x-y|)^\lambda}, \ x \in \mathbb{R}^n,$$

where $\varphi_j(\lambda) = \varphi(2^{-j}\lambda)$ and $f \in \mathcal{S}'$. Then it is clear that

$$\varphi^*_{j,\lambda}(\sqrt{L}) f(x) \geq |\varphi_j(\sqrt{L}) f(x)|, \ x \in \mathbb{R}^n.$$

In addition, for $s, \lambda > 0,$ we set

$$\varphi^*_{\lambda}(s\sqrt{L}) f(x) = \sup_{y\in\mathbb{R}^n} \frac{|\varphi(s\sqrt{L}) f(y)|}{(1+|x-y|/s)^\lambda}, \ f \in \mathcal{S}'.$$

Remark 2.11. Due to (2.4), $\varphi^*_\lambda(s\sqrt{L}) f(x) < \infty$ for all $x \in \mathbb{R}^n$, provided that λ is sufficiently large.

In what follows, by a "partition of unity", we mean a function $\psi \in S(\mathbb{R})$ with $\operatorname{supp}\psi \subset [1/2, 2]$

$$\int \frac{\psi(s)}{s} ds \neq 0 \text{ and } \sum_{j\in\mathbb{Z}} \psi_j(\lambda) = 1, \lambda \in (0,\infty), \text{ where } \psi_j(\lambda) = \psi(2^{-j}\lambda), j \in \mathbb{Z}.$$

Proposition 2.12. [Bui et al., 2020, Proposition 2.16]
Let $\psi \in S(\mathbb{R})$ *with* $\operatorname{supp}\psi \subset [1/2, 2]$ and $\varphi \in S(\mathbb{R})$ *be a partition of unity. Then for any* $\lambda > 0, j \in \mathbb{Z}$, *we have for all* $f \in \mathcal{S}'_\infty$ *and* $x \in \mathbb{R}^n$*:*

$$\sup_{s\in[2^{-j-1},2^{-j}]} \psi^*_\lambda(s\sqrt{L}) f(x) \lesssim \sum_{k=j-2}^{j+3} \varphi^*_{k,\lambda}(\sqrt{L}) f(x). \tag{2.5}$$

Proposition 2.13. [Bui et al., 2020, Proposition 2.17]

Let ψ *be a partition of unity. Then for* $\lambda, s, r > 0$, *we have for all* $f \in \mathcal{S}'_\infty$ *and* $x \in \mathbb{R}^n$*:*

$$\psi^*_\lambda(s\sqrt{L}) f(x) \lesssim \left[\int_{\mathbb{R}^n} s^{-n} |\psi(s\sqrt{L}) f(z)|^r \left(1 + \frac{|x-z|}{s}\right)^{-\lambda r} dz \right]^{1/r}. \tag{2.6}$$

Proposition 2.14. [Bui et al., 2020, Proposition 2.18]
Let ψ *be a partition of unity and* $\varphi \in S(\mathbb{R})$ *be an even function such that* $\varphi \neq 0$ *on* $[1/2, 2]$. *Then for any* $\lambda, r > 0$ *and* $j \in \mathbb{Z}$ *we have for all* $f \in \mathcal{S}'$:

$$|\psi_j(\sqrt{L}) f(x)| \lesssim \left(\int_{2^{-j-2}}^{2^{-j+2}} |\varphi^*_\lambda(s\sqrt{L}) f(x)|^r \frac{ds}{s}\right)^{1/r}.$$

3. Triebel-Lizorkin-Morrey spaces associated with *L*

3.1. Definitions of TLM spaces associated with L

Definition 3.1. Let ψ be a partition of unity. For $0 < p \leq r < \infty, 0 < q \leq \infty, \alpha \in \mathbb{R}$, the homogeneous TLM space $FM^{\alpha,\psi,L}_{p,q,r}$ is defined by

$$FM^{\alpha,\psi,L}_{p,q,r} = \{ f \in \mathcal{S}'_\infty : \| f \|_{FM^{\alpha,\psi,L}_{p,q,r}} < \infty \},$$

where

$$\| f \|_{FM^{\alpha,\psi,L}_{p,q,r}} = \Big\| \Big[\sum_{j\in\mathbb{Z}} (2^{j\alpha} | \psi_j\big(\sqrt{L}\big) f |)^q \Big]^{1/q} \Big\|_{M^r_p} .$$

Note that $\psi_j(\sqrt{L}) f = 0$ for all $j \in \mathbb{Z}$ if and only if $f \in \mathbb{P}$ (see page 27 of Bui et al., 2020). Hence, each of the above spaces is a quasi-normed linear space, particularly a normed linear space when $p, q \geq 1$.

In light of Proposition 2.12, one has:

Proposition 3.2. *Let* ψ, φ *be partitions of unity and assume* $\operatorname{supp}\psi, \operatorname{supp}\varphi \subset [1/2, 2]$, $0 < p \leq r < \infty$, $0 < q \leq \infty$, $\alpha \in \mathbb{R}$ *and* $\lambda > 0$. *Then the following norm equivalence holds for all* $f \in S'_\infty$: $\Big\| \Big[\sum_{j\in\mathbb{Z}} (2^{j\alpha} | \psi^*_{j,\lambda}(\sqrt{L}) f |)^q \Big]^{1/q} \Big\|_{M^r_p} \sim \Big\| \Big[\sum_{j\in\mathbb{Z}} (2^{j\alpha} | \varphi^*_{j,\lambda}(\sqrt{L}) f |)^q \Big]^{1/q} \Big\|_{M^r_p}$.

We next prove the following result.

Proposition 3.3. *Let* ψ *be a partition of unity. Then for* $0 < p \leq r < \infty$, $0 < q \leq \infty$, $\alpha \in \mathbb{R}$ *and* $\lambda > \max\{n/p, n/q\}$, *we have:* $\Big\| \Big[\sum_{j\in\mathbb{Z}} (2^{j\alpha} | \psi^*_{j,\lambda}(\sqrt{L}) f |)^q \Big]^{1/q} \Big\|_{M^r_p} \sim \| f \|_{FM^{\alpha,\psi,L}_{p,q,r}}$.

Proof. In view of Proposition 3.2, it suffices to prove that

$$\Big\| \Big[\sum_{j\in\mathbb{Z}} (2^{j\alpha} | \psi^*_{j,\lambda}(\sqrt{L}) f |)^q \Big]^{1/q} \Big\|_{M^r_p} \lesssim \Big\| \Big[\sum_{j\in\mathbb{Z}} (2^{j\alpha} | \psi_j(\sqrt{L}) f |)^q \Big]^{1/q} \Big\|_{M^r_p} . \qquad (3.7)$$

Indeed, taking $\theta < \min\{p, q\}$ so that $\lambda > n/\theta$, then applying (2.6) gives

$$\psi^*_{j,\lambda}(\sqrt{L}) f(x) \lesssim \Big[\int_{\mathbb{R}^n} 2^{jn} | \psi_j(\sqrt{L}) f(z) |^\theta (1 + 2^j | x - z |)^{-\lambda\theta} dz \Big]^{1/\theta} \lesssim \mathcal{M}_\theta(| \psi_j(\sqrt{L}) f |)(x),$$

where we use Lemma 2.1 in the last inequality. The desired inequality (3.7) then follows by the Fefferman-Stein maximal inequality (2.1).

As a consequence of Proposition 3.2 and Proposition 3.3, we obtain the following theorem.

Theorem 3.4. *Let* ψ *and* φ *be partitions of unity. Then the spaces* $FM^{\alpha,\psi,L}_{p,q,r}$ *and* $FM^{\alpha,\varphi,L}_{p,q,r}$ *coincide with equivalent norms for all* $0 < p \leq r < \infty$, $0 < q \leq \infty$, $\alpha \in \mathbb{R}$. *For this reason, we define the spaces* $FM^{\alpha,L}_{p,q,r}$ *to be any spaces* $FM^{\alpha,\psi,L}_{p,q,r}$ *with any partitions of unity* ψ.

Remark 3.5. It is standard to show that the space $FM^{\alpha,L}_{p,q,r}$ is complete and is continuously embedded into S'_∞.

3.2. Continuous characterizations by functions with compact supports

In this subsection, we will prove continuous characterizations for new TLM spaces including the ones using Lusin functions and the Littlewood-Paley functions.

Theorem 3.6. *Let* ψ *be a partition of unity. Then for* $0 < p \le r < \infty$, $0 < q \le \infty$, $\alpha \in \mathbb{R}$ *and* $\lambda > \max\{n/p, n/q\}$, *we have for all* $f \in S'_\infty$ *:*

$$\| f \|_{FM^{\alpha,L}_{p,q,r}} \sim \Big\| \Big(\int_0^\infty [t^{-\alpha} | \psi(t\sqrt{L}) f |]^q \frac{dt}{t}\Big)^{1/q} \Big\|_{M^r_p} \sim \Big\| \Big(\int_0^\infty [t^{-\alpha} \psi^*_\lambda(t\sqrt{L}) f]^q \frac{dt}{t}\Big)^{1/q} \Big\|_{M^r_p}.$$

Proof. The proof is divided into three steps.

Step 1. We first claim that

$$\Big\| \Big(\int_0^\infty [t^{-\alpha} | \psi(t\sqrt{L}) f |]^q \frac{dt}{t}\Big)^{1/q} \Big\|_{M^r_p} \lesssim \| f \|_{FM^{\alpha,L}_{p,q,r}}. \tag{3.8}$$

Indeed, for $t \in [2^{-j-1}, 2^{-j}]$ with $j \in \mathbb{Z}$, it follows from (2.5) that

$$\sup_{t\in[2^{-j-1},2^{-j}]} | \psi(t\sqrt{L}) f(x) | \lesssim \sum_{k=j-2}^{j+3} \psi^*_{k,\lambda}(\sqrt{L}) f(x).$$

The estimate (3.8) then follows from the above inequality and Proposition 3.2.

Step 2. We next prove that

$$\| f \|_{FM^{\alpha,L}_{p,q,r}} \lesssim \Big\| \Big(\int_0^\infty [t^{-\alpha} \psi^*_\lambda(t\sqrt{L}) f]^q \frac{dt}{t}\Big)^{1/q} \Big\|_{M^r_p}. \tag{3.9}$$

Indeed, in view of Proposition 2.14, we derive

$$| \psi_j(\sqrt{L}) f(x) | \lesssim \Big(\int_{2^{-j-2}}^{2^{-j+2}} |\psi^*_\lambda(s\sqrt{L}) f(x)|^q \frac{ds}{s}\Big)^{1/q},$$

which implies (3.9).

Step 3. We complete the proof of Theorem 3.6 by showing that

$$\Big\| \Big(\int_0^\infty [t^{-\alpha} \psi^*_\lambda(t\sqrt{L}) f]^q \frac{dt}{t}\Big)^{1/q} \Big\|_{M^r_p} \lesssim \Big\| \Big(\int_0^\infty [t^{-\alpha} | \psi(t\sqrt{L}) f |]^q \frac{dt}{t}\Big)^{1/q} \Big\|_{M^r_p}. \tag{3.10}$$

Taking $\theta < \min\{p,q\}$ so that $\lambda > n/\theta$, then applying (2.6) yields that for all $t \in [1,2]$:

$$| \psi^*_\lambda(2^{-j} t\sqrt{L}) f(x) |^\theta \lesssim \int_{\mathbb{R}^n} 2^{jn} | \psi(2^{-j} t\sqrt{L}) f(z) |^\theta (1 + 2^j | x - z |)^{-\lambda\theta} dz.$$

Since $\theta < q$, we use Minkowski's inequality to obtain

$$\Big(\int_1^2 |\psi^*_\lambda(2^{-j} t\sqrt{L}) f(x)|^q \frac{dt}{t}\Big)^{\theta/q} \lesssim \int_{\mathbb{R}^n} 2^{jn} \Big(\int_1^2 |\psi(2^{-j} t\sqrt{L}) f(z)|^q \frac{dt}{t}\Big)^{\theta/q} (1 + 2^j | x - z |)^{-\lambda\theta} dz.$$

By a change of variables, it is clear to see that

$$\left[\int_{2^{-j}}^{2^{-j+1}} (t^{-\alpha} | \psi^*_\lambda(t\sqrt{L}) f(x) |)^q \frac{dt}{t}\right]^{\theta/q}$$

$$\lesssim \int_{\mathbb{R}^n} 2^{jn} \left[\int_{2^{-j}}^{2^{-j+1}} (t^{-\alpha} | \psi(t\sqrt{L}) f(z) |)^q \frac{dt}{t}\right]^{\theta/q} (1 + 2^j | x - z |)^{-\lambda\theta} dz.$$

At this stage, in light of Lemma 2.1, we deduce that if $\lambda\theta > n$ then

$$\Big(\int_{2^{-j}}^{2^{-j+1}} |\psi_\lambda^*(t\sqrt{L})f(x)|^q \frac{dt}{t}\Big)^{1/q} \lesssim \mathcal{M}_\theta\left[\Big(\int_{2^{-j}}^{2^{-j+1}} |\psi(t\sqrt{L})f|^q \frac{dt}{t}\Big)^{1/q}\right](x),$$

which, combined with (2.1), gives the desired estimate (3.10).

4. Atomic decompositions for TLM spaces $FM_{p,q,r}^{\alpha,L}$

In this section, we prove the atomic decomposition characterization for TLM spaces. We first present the definition of atoms related to L.

Definition 4.1. Let $0 < r \le \infty$ and $M \in \mathbb{N}_+$. A function a is said to be an (L, M, r) atom if there exists a dyadic cube $Q \in \mathcal{D}$ such that:

i. $a = L^M b$ with $b \in D(L^M)$, where $D(L^M)$ is the domain of L^M;

ii. $\operatorname{supp} L^k b \subset 3Q,\ k = 0,\dots,2M$;

iii. $|L^k b(x)| \le \ell(Q)^{2(M-k)} |Q|^{-1/r}, k = 0,\dots,2M$.

Under the proof of [Bui et al., 2020, Theorem 4.2], we have the following lemma.

Lemma 4.2. *Let ψ be a partition of unity, Φ be a function as in Lemma 2.5, $0 < p \le r < \infty$, $0 < q \le \infty$, $\alpha \in \mathbb{R}$, and $M \in \mathbb{N}_+$. Set $\psi_M(\theta) = \theta^{-2M}\psi(\theta)$, then for any $f \in S'_\infty$, we have the following statements:*

i. $f = \sum_{\nu\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_\nu} s_Q a_Q$ in $\mathcal{S}'_\infty$, *where* $s_Q = |Q|^{1/r} \sup_{y\in Q} \int_{2^{-\nu-1}}^{2^{-\nu}} |\psi_M(t\sqrt{L}) f(y)| \frac{dt}{t}$, $a_Q = L^M b_Q$

are (L, M, r) atoms, and $b_Q = \dfrac{1}{s_Q}\int_{2^{-\nu-1}}^{2^{-\nu}} t^{2M}\Phi(t\sqrt{L})[\psi_M(t\sqrt{L}) f.1_Q]\dfrac{dt}{t}.$ *(4.11)*

ii. For any $\lambda > 0, m > \alpha/2$, we have: $\sum_{Q\in\mathcal{D}_\nu} |Q|^{-1/r} |s_Q| 1_Q \lesssim \sum_{j\in\mathbb{Z}} 2^{-2m|\nu-j|}\psi^*_{j,\lambda}(\sqrt{L}) f.$

Next, we prove the following atomic decomposition theorem for the spaces $FM_{p,q,r}^{\alpha,L}$.

Theorem 4.3. *Let $0 < p \le r < \infty$, $0 < q \le \infty$, $\alpha \in \mathbb{R}$, and $M \in \mathbb{N}_+$. If $f \in FM_{p,q,r}^{\alpha,L}$ then there exist a sequence of (L, M, r) atoms $(a_Q)_{Q\in\mathcal{D}_\nu,\nu\in\mathbb{Z}}$ and a sequence of coefficients $(s_Q)_{Q\in\mathcal{D}_\nu,\nu\in\mathbb{Z}}$ so that:* $f = \sum_{\nu\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_\nu} s_Q a_Q$ in $\mathcal{S}'_\infty$. *Moreover, one has*

$$\Big\|\Big[\sum_{\nu\in\mathbb{Z}} 2^{\nu\alpha q}\Big(\sum_{Q\in\mathcal{D}_\nu} |Q|^{-1/r} |s_Q| 1_Q\Big)^q\Big]^{1/q}\Big\|_{M_p^r} \lesssim \|f\|_{FM_{p,q,r}^{\alpha,L}}.$$

Proof. Let ψ be a partition of unity, Φ be a function as in Lemma 2.5. *Set* $\psi_M(\theta) = \theta^{-2M}\psi(\theta)$. In light of Lemma 4.2, for $f \in S'_\infty$, we have:

$f = \sum_{\nu\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_\nu} s_Q a_Q$ in $\mathcal{S}'_\infty$, where $s_Q = |Q|^{1/r}\sup_{y\in Q}\int_{2^{-\nu-1}}^{2^{-\nu}} |\psi_M(t\sqrt{L})f(y)|\frac{dt}{t}$, $a_Q = L^M b_Q$ are (L,M,r) atoms, and $b_Q = \frac{1}{s_Q}\int_{2^{-\nu-1}}^{2^{-\nu}} t^{2M}\Phi(t\sqrt{L})[\psi_M(t\sqrt{L})f.1_Q]\frac{dt}{t}$.

Moreover, for any $\lambda > 0, m > \alpha/2$, we have $\sum_{Q\in\mathcal{D}_\nu} |Q|^{-1/r}|s_Q|1_Q \lesssim \sum_{j\in\mathbb{Z}} 2^{-2m|\nu-j|}\psi^*_{j,\lambda}(\sqrt{L})f$.

Hence

$$\Big\|\Big[\sum_{\nu\in\mathbb{Z}} 2^{\nu\alpha q}\Big(\sum_{Q\in\mathcal{D}_\nu}|Q|^{-1/r}|s_Q|1_Q\Big)^q\Big]^{1/q}\Big\|_{M^r_p} \lesssim \Big\|\Big[\sum_{\nu\in\mathbb{Z}} 2^{\nu\alpha q}\Big(\sum_{j\in\mathbb{Z}} 2^{-2m|\nu-j|}\psi^*_{j,\lambda}(\sqrt{L})f\Big)^q\Big]^{1/q}\Big\|_{M^r_p}$$

$$\lesssim \Big\|\Big[\sum_{\nu\in\mathbb{Z}}\Big(\sum_{j\in\mathbb{Z}} 2^{-2m|\nu-j|+\alpha(\nu-j)} 2^{j\alpha}\psi^*_{j,\lambda}(\sqrt{L})f\Big)^q\Big]^{1/q}\Big\|_{M^r_p}.$$

At this stage, we apply Young's inequality when $q > 1$ and the inequality $(\sum_j |a_j|)^q \le \sum_j |a_j|^q$ when $0 < q \le 1$ to deduce that

$$\Big\|\Big[\sum_{\nu\in\mathbb{Z}} 2^{\nu\alpha q}\Big(\sum_{Q\in\mathcal{D}_\nu}|Q|^{-1/r}|s_Q|1_Q\Big)^q\Big]^{1/q}\Big\|_{M^r_p} \lesssim \Big\|\Big[\sum_{j\in\mathbb{Z}}\Big(2^{j\alpha}\psi^*_{j,\lambda}(\sqrt{L})f\Big)^q\Big]^{1/q}\Big\|_{M^r_p} \lesssim \|f\|_{FM^{\alpha,L}_{p,q,r}},$$

where we use Proposition 3.3 in the last inequality. This completes our proof.

For the converse direction, we obtain the following theorem.

Theorem 4.4. *Let* $0 < p \le r < \infty$, $0 < q \le \infty$, $\alpha \in \mathbb{R}$, *and* $M \in \mathbb{N}_+$. *If*

$$f = \sum_{\nu\in\mathbb{Z}}\sum_{Q\in\mathcal{D}_\nu} s_Q a_Q \text{ in } \mathcal{S}'_\infty,$$

where $(a_Q)_{Q\in\mathcal{D}_\nu,\nu\in\mathbb{Z}}$ *is a sequence of* (L,M,r) *atoms and* $(s_Q)_{Q\in\mathcal{D}_\nu,\nu\in\mathbb{Z}}$ *is a sequence of coefficients satisfying* $\Big\|\Big[\sum_{\nu\in\mathbb{Z}} 2^{\nu\alpha q}\Big(\sum_{Q\in\mathcal{D}_\nu}|Q|^{-1/r}|s_Q|1_Q\Big)^q\Big]^{1/q}\Big\|_{M^r_p} < \infty$, *then* $f \in FM^{\alpha,L}_{p,q,r}$ *and*

$$\|f\|_{FM^{\alpha,L}_{p,q,r}} \lesssim \Big\|\Big[\sum_{\nu\in\mathbb{Z}} 2^{\nu\alpha q}\Big(\sum_{Q\in\mathcal{D}_\nu}|Q|^{-1/r}|s_Q|1_Q\Big)^q\Big]^{1/q}\Big\|_{M^r_p},$$

provided that $M > \frac{n}{2} + \frac{1}{2}\max\left\{\alpha, \frac{n}{1\wedge r\wedge q} - \alpha\right\}$.

Proof. Fix $\tilde{q} \in (1,\infty)$ and $\theta < \min\{1,p,q\}$. It follows from the proof of [Bui et al., 2020, Theorem 4.7] that

$$2^{j\alpha}|\psi_j(\sqrt{L})f| \lesssim \sum_{\nu:\nu\ge j} 2^{-(\nu-j)(2M-n\tilde{q}/\theta-\alpha)}\mathcal{M}_\theta\Big(\sum_{Q\in\mathcal{D}_\nu} 2^{\nu\alpha}|s_Q||Q|^{-1/r}1_Q\Big)$$

$$+\sum_{\nu:\nu<j} 2^{-(2M-\alpha)(j-\nu)}\mathcal{M}_\theta\Big(\sum_{Q\in\mathcal{D}_\nu} 2^{\nu\alpha}|s_Q||Q|^{-1/r}1_Q\Big).$$

By using (2.2), we conclude that

$$\|f\|_{FM^{\alpha,L}_{p,q,r}} = \Big\|\Big[\sum_{j\in\mathbb{Z}}(2^{j\alpha}|\psi_j(\sqrt{L})f|)^q\Big]^{1/q}\Big\|_{\mathrm{M}^r_p} \lesssim \Big\|\Big[\sum_{\nu\in\mathbb{Z}}2^{\nu\alpha q}\Big(\sum_{Q\in\mathcal{D}_\nu}|Q|^{-1/r}|s_Q|1_Q\Big)^q\Big]^{1/q}\Big\|_{\mathrm{M}^r_p},$$

which completes the proof.

Remark 4.5. By a careful examination of the proof of Theorem 4.3, it is apparent to see that each atom $a_Q = L^{2M}b_Q$, defined by (4.11), belongs to the space of test functions S_∞. As a direct consequence of the atomic decomposition results mentioned above, the space S_∞ of test functions is dense in $FM^{\alpha,L}_{p,q,r}$ for $0 < p \le r < \infty$, $0 < q \le \infty$, $\alpha \in \mathbb{R}$.

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KHÔNG GIAN TRIEBEL-LIZORKIN-MORREY
LIÊN KẾT VỚI TOÁN TỬ LIÊN HỢP KHÔNG ÂM

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TÓM TẮT

Xét L là một toán tử liên hợp không âm trên $L^2(\mathbb{R}^n)$ sao cho nhân nhiệt của L thỏa mãn điều kiện bị chặn trên Gaussian. Trong bài báo này, chúng tôi giới thiệu không gian Triebel-Lizorkin-Morrey $FM^{\alpha,\mathrm{L}}_{p,q}(\mathbb{R}^n)$ liên kết với toán tử L, trong đó $0 < p, q \leq \infty, \alpha \in \mathbb{R}$. Chúng tôi chứng minh rằng các không gian mới này thỏa mãn các đặc trưng quan trọng như đặc trưng liên tục theo các hàm bình phương hoặc đặc trưng phân tích nguyên tử.

Từ khóa: phân tích nguyên tử; đặc trưng liên tục; điều kiện bị chặn trên Gaussian; không gian Triebel-Lizorkin-Morrey