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Research Article

CALDERÓN-ZYGMUND COMMUTATORS OF TYPE THETA ON GENERALIZED MORREY-LORENTZ SPACE

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ABSTRACT

In this paper, we consider Calderón-Zygmund commutators $[b,T]$ of type θ (see Definitions 1.3, 1.4, and 1.5 in Section 1) on generalized Morrey-Lorentz spaces $\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$ (see Definition 1.1). In this setting, we first establish the pointwise estimates for the Hardy-Littlewood maximal operator and the sharp maximal operator acting on Calderón-Zygmund operators of the type θ and Calderón-Zygmund commutators of type θ (see Lemma 2.4, Lemma 2.5 in Section 2) by using Kolmogorov's inequality, Holder's inequality, the conditions of standard kernels in the definition of Calderón-Zygmund operators of type θ, and the well-known consequence of John-Nirenberg inequality. Thanks to these significant pointwise estimates, we then prove that Calderón-Zygmund operators of type θ are bounded on generalized Morrey-Lorentz spaces $\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$ (see Theorem 2.1) by modifying the ideas and techniques related to maximal operators proposed by Thai et al. (2022a, 2022b), Carro et al. (2021), and Liu et al. (2002). Futhermore, we deduce that commutators $[b,T]$ of type θ are also bounded on these spaces due to the pointwise estimate for the sharp maximal operator acting on commutators $[b,T]$ of type θ and the boundedness of T in $\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$.

Keywords: Calderón-Zygmund commutator of type θ; generalized Morrey-Lorentz space; maximal operator

1. Introduction

Calderón-Zygmund operators, which were first introduced by Coifman et al. (1978), have played an important role in modern harmonic analysis. Since then, there have been a number of studies about the boundedness of Calderón-Zygmund operators and their commutators. Grafakos (2009) indicated that Calderón-Zygmund operators are bounded on L^p for $1 < p < \infty$ and from L^1 to $L^{1,\infty}$. Carro et al. (2021) found sufficient conditions for a

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pair of weights u and w such that the Calderón-Zygmund operator T and its commutator $\left[b,T\right]$, with $b\in BMO$, is bounded on weighted Lorentz spaces $\Lambda_u^p(w)$, for $1<p<\infty$. Recently, Dao et al. (2021) showed the Lorentz boundedness of Calderón-Zygmund operator T and its commutator $\left[b,T\right]$ on spaces of homogeneous type.

In this paper, we consider generalized Morrey-Lorentz spaces in $\mathbb{R}^n$ with Lebesgue measure. We assume that $p\in\left(1,\infty\right)$, $r\in\left[1,\infty\right)$ and the function $\varphi(t):\left(0;\infty\right)\to\left(0;\infty\right)$ satisfies the following conditions:

$$\begin{cases} i)\ \varphi(t)\text{ is nonincreasing,}\\ ii)\ \left|B_t\right|\varphi^p(t)\text{ is nondecreasing, for any ball } B_t\subset X,\\ iii)\ \varphi(2t)\le D\varphi(t),\ \forall t>0,\end{cases} \tag{1.1}$$

for some constant $0<D<1$.

Now, we define the generalized Morrey – Lorentz space as follows:

Definition 1.1. *A real-value function f on $\mathbb{R}^n$ is said to belong to the generalized Morrey-Lorentz space $\mathbf{M}_\varphi^{p,r}\left(\mathbb{R}^n\right)$ if the following norm is finite:*

$$\|f\|_{\mathbf{M}_\varphi^{p,r}}=\sup_{B(x,t)}\frac{\|f\|_{L^{p,r}\left(B(x,t)\right)}}{\left|B\left(x,t\right)\right|^{1/p}\varphi(t)}, \tag{1.2}$$

where the supremum is taken over all the balls $B\left(x,t\right)$ in $\mathbb{R}^n$, and $\|f\|_{L^{p,r}\left(B(x,t)\right)}$ denotes the Lorentz norm of f on $B\left(x,t\right)$ (see Dao et al., 2021, Definition 2.5).

Definition 1.2. *A function $b\in L^1_{loc}\left(\mathbb{R}^n\right)$ is said to belong to the space $BMO\left(\mathbb{R}^n\right)$ if*

$$\|b\|_{BMO\left(\mathbb{R}^n\right)}:=\sup_B\frac{1}{|B|}\int_B\left|b(x)-b_B\right|dx<\infty,$$

where

$$b_B=\frac{1}{|B|}\int_B b(x)dx,$$

and the supremum is taken over all balls $B\subset\mathbb{R}^n$.

On the other hand, Yabuta (1985) first introduced Calderón-Zygmund operators T of type θ and obtained the boundedness of T on Lebesgue spaces. After that, Liu et al. (2002) show that if $b\in BMO\left(\mathbb{R}^n\right)$ and T is a Calderón-Zygmund operator of type θ then $\left[b,T\right]$ is bounded from $H^1\left(\mathbb{R}^n\right)$ to weak $L^1\left(\mathbb{R}^n\right)$. More recently, Thai et al. (2022) proved the boundedness of Calderón-Zygmund operators and commutators of type θ on the generalized weighted Lorentz spaces $\Lambda_u^p(w)$. Inspired by the above works, we aim to study the boundedness of Calderón-Zygmund operators and commutators on the generalized Morrey-Lorentz space $\mathbf{M}_{\varphi}^{p,r}\left(\mathbb{R}^n\right)$.

The definition of the Calderón-Zygmund operator T of type θ and its commutator is presented below:

Definition 1.3. *(Yabuta, 1985) Let θ be a nonnegative, nondecreasing function on $(0,\infty)$ with*

$$\int_0^1 \theta(t)t^{-1}dt < \infty. \qquad (1.3)$$

A continuous function $K(x,y)$ on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x): x \in \mathbb{R}^n\}$ is said to be a standard kernel of type θ if it satisfies the following conditions.

(i) (Size condition)

$$|K(x,y)| \le \frac{C}{|x-y|^n}. \qquad (1.4)$$

(ii) (Regularity condition)

$$|K(x,y)-K(x_0,y)| + |K(y,x)-K(y,x_0)| \le C|x_0-y|^{-n}\,\theta\left(\frac{|x_0-x|}{|y-x_0|}\right), \qquad (1.5)$$

for every x, x_0, y with $2|x-x_0| < |y-x_0|$.

Definition 1.4. *(Yabuta, 1985) Let θ be a function as in Definition 1.3. A linear operator T from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ is said to be a Calderón-Zygmund operator of type θ if it satisfies the following conditions.*

(i) T is bounded on $L^2(\mathbb{R}^n)$, which means

$$\|Tf\|_{L^2} \le C\|f\|_{L^2}, \quad \text{for every } f \in C_0^\infty(\mathbb{R}^n). \qquad (1.6)$$

(ii) There exists a standard kernel K of type θ such that for every function $f \in C_0^\infty(\mathbb{R}^n)$ and $x \notin supp(f)$

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)dy. \qquad (1.7)$$

Definition 1.5. *Let T be a Calderón-Zygmund operator of type θ and $b \in L^1_{loc}\left(\mathbb{R}^n\right)$. Then the commutator $\left[b,T\right]$ is defined by*

$$\left[b,T\right]f(x) := b(x)T(f)(x) - T(bf)(x)$$

for suitable functions f.

The structure of this paper is as follows. We establish pointwise estimates for sharp maximal operators and key lemmas in Section 2. Then, we prove the boundedness of the Calderón-Zygmund operator of type θ and its commutator on $\mathbf{M}_{\varphi}^{p,r}\left(\mathbb{R}^n\right)$.

As usual, for any $q \in [1, \infty)$, we denote by q' the conjugate exponent of q, that is, $\frac{1}{q} + \frac{1}{q'} = 1$. We also denote a constant by C, which only depends on p, q, r, n, φ and may change at different lines. In addition, we write $A \lesssim B$ if there exists a constant $C > 0$ such that $A \le CB$. Finally, we write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$.

2. Main results

2.1. Pointwise estimates for sharp maximal operators

For $q > 0$, let M_q be the modified Hardy–Littlewood maximal function

$$M_q f(x) = M(|f|^q)^{1/q}(x) = \left(\sup_{r>0} \frac{1}{|B|} \int_B |f(y)|^q \, dy \right)^{1/q},$$

and let $M_\beta^\#$ be the modified sharp maximal function

$$M_q^\# f(x) = \sup_{r>0} \inf_{c \in \mathbb{R}} \left(\frac{1}{|B|} \int_B \left| |f(y)|^q - |c|^q \right| dy \right)^{1/q},$$

where $B = B(x, r)$ is a ball in $\mathbb{R}^n$.

We briefly write $M^\#$ for $M_1^\#$.

Remark 2.1. *It is clear to see that*

$$M_q^\#(f)(x) \sim \sup_{r>0} \left(\frac{1}{|B|} \int_B \left| |f(y)|^q - |f|_B^q \right| dy \right)^{1/q}. \tag{2.1}$$

Lemma 2.1. (The Kolmogorov's inequality) *(Lu et al., 2007, Theorem 1.3.3) Suppose that T is a sublinear operator from $L^p(\mathbb{R}^n)$, $1 \le p < \infty$, to the space of measurable functions on $\mathbb{R}^n$.*

(i) If T is of weak type (p, p), then for all $0 < r < p$ and all sets E with finite measure, there exists a constant $C > 0$ such that

$$\frac{1}{|E|} \int_E |Tf(x)|^r \, dx \le C \frac{1}{|E|^{r/p}} \| f \|_p^r. \tag{2.2}$$

(ii) If there exist $r \in (0, p)$ and a constant $C > 0$ such that (2.2) holds for all sets E with finite measure and $f \in L^p(\mathbb{R}^n)$, then T is of weak type (p, p).

Lemma 2.2. *Let $p \in (1, \infty)$, $r \in [1, \infty)$, and let $\varphi(t)$ satisfy (1.1). Then, for any $0 < q < p$ there is a positive constant $C = C(p, q, r)$ such that*

$$\left\| M_q(f) \right\|_{\mathbf{M}_\varphi^{p,r}} \le C \| f \|_{\mathbf{M}_\varphi^{p,r}}.$$

Proof. Let $B_t = B(x, t)$ be any ball in $\mathbb{R}^n$. Then we write

$$f = f1_{B_t} + f1_{B_t^c} := f_1 + f_2.$$

So, $M_q(f)(x) \le M_q(f_1)(x) + M_q(f_2)(x)$. First, we estimate $M_q(f_1)$. Since M_q maps $L^{p,r}(\mathbb{R}^n) \to L^{p,r}(\mathbb{R}^n)$ (see Dao et al., 2021), then we get

$$\frac{1}{|B_t|^{1/p}\varphi(t)}\|M_q(f_1)\|_{L^{p,r}(B_t)} \lesssim \frac{1}{|B_t|^{1/p}\varphi(t)}\|f_1\|_{L^{p,r}(\mathbb{R}^n)} = \frac{1}{|B_t|^{1/p}\varphi(t)}\|f\|_{L^{p,r}(B_t)} \le \|f\|_{\mathbf{M}_\varphi^{p,r}},$$

which yields

$$\|M_q(f_1)\|_{\mathbf{M}_\varphi^{p,r}} \le \|f\|_{\mathbf{M}_\varphi^{p,r}}. \qquad (2.3)$$

Next, since $f_2 = 0$ on B_t, then we observe that for any $z \in B_{t/2}$,

$$M_q(f_2)(z) \le \sup_{z\in B_\delta, \delta > t/8} \left(\frac{1}{|B_\delta|}\int_{B_\delta} |f(y)|^q \, d\mu(y) \right)^{1/q}.$$

Then, it follows from Holder's inequality in Lorentz spaces and the monotonicity of φ that

$$M_q(f_2)(z) \lesssim \sup_{z\in B_\delta, \delta > t/8} \left\{ |B_\delta|^{-1/q} \|f\|_{L^{p,r}(B_\delta)} |B_\delta|^{1/q - 1/p} \right\} = \sup_{z\in B_\delta, \delta > t/8} \left\{ |B_\delta|^{-1/p} \|f\|_{L^{p,r}(B_\delta)} \right\}$$

$$\lesssim \sup_{z\in B_\delta, \delta > t/8} \left\{ \varphi(\delta) \|f\|_{\mathbf{M}_\varphi^{p,r}} \right\} \lesssim \varphi(t/8)\|f\|_{\mathbf{M}_\varphi^{p,r}}.$$

Therefore, we deduce that

$$\|M_q(f_2)\|_{L^{p,r}(B_{t/8})} \lesssim |B_{t/8}|^{\frac{1}{p}} \varphi(t/8)\|f\|_{\mathbf{M}_\varphi^{p,r}}$$

which implies

$$\|M_q(f_2)\|_{\mathbf{M}_\varphi^{p,r}} \lesssim \|f\|_{\mathbf{M}_\varphi^{p,r}}. \qquad (2.4)$$

By combining (2.3) and (2.4), we complete the proof of Lemma 2.2.

In the sequel, we use the following lemma.

Lemma 2.3. *(Tran, 2014, Lemma 3.3) Given balls* $B_1 = B(x_1, r_1)$, $B_2 = B(x_2, r_2)$, *whose intersection is not empty and* $\frac{1}{2}r_2 \le r_1 \le 2r_2$. *Then for each function* $b \in BMO(\mathbb{R}^n)$, *we have* $|b_{B_1} - b_{B_2}| \le 2C\|b\|_{BMO}$.

To establish the main results of this stuy, we need to prove the following pointwise estimates for the modified sharp maximal operator, following Thai et al (2022a, 2022b) and Liu et al. (2002).

Lemma 2.4. *Let* T *be a Calderón-Zygmund operator of type* θ *and* $1 < q < p$. *Then, there exists a constant* $C > 0$ *such that*

$$M^{\#}(Tf)(x_0) \le CM_q f(x_0),$$

for any $f \in \mathbf{M}_{\varphi}^{p,r}\left(\mathbb{R}^n\right)$ *and* $x_0 \in \mathbb{R}^n$.

Proof. For any ball $B = B(x_0, r)$ and any $f \in \mathbf{M}_{\varphi}^{p,r}\left(\mathbb{R}^n\right)$, we write $f = f_1 + f_2$, with $f_1 = f\chi_{2B}$ and $f_2 = f\chi_{\mathbb{R}^n \setminus 2B}$. Then it is sufficient to prove that there exists a positive constant C such that

$$\left(\frac{1}{|B|}\int_B \left||Tf| - \left(Tf_2\right)_B\right| dx\right) \le CM_q f(x_0).$$

Indeed, we have

$$\frac{1}{|B|}\int_B \left||Tf| - |\left(Tf_2\right)_B|\right| dx \le \frac{1}{|B|}\int_B \left|Tf - \left(Tf_2\right)_B\right| dx = \frac{1}{|B|}\int_B \left|Tf_1 + \left(Tf_2 - (Tf_2)_B\right)\right| dx$$
$$\le \frac{1}{|B|}\int_B |Tf_1|\, dx + \frac{1}{|B|}\int_B |Tf_2 - \left(Tf_2\right)_B|\, dx.$$

Set $I_1 = \frac{1}{|B|}\int_B |Tf_1|\, dx$ and $I_2 = \frac{1}{|B|}\int_B |Tf_2 - \left(Tf_2\right)_B|\, dx$.

Since Yabuta (1985) showed that T is of strong type (q,q) on $L^q(\mathbb{R}^n)$, so T is of weak type (q,q) on $L^q(\mathbb{R}^n)$. Therefore, based on the Kolmogorov's inequality, we get

$$I_1 \lesssim \frac{1}{|B|}\int_B |Tf_1|\, dx \lesssim \frac{1}{|B|^{1/q}}\left(\int_{\mathbb{R}^n} |f_1^q(x)|\, dx\right)^{1/q},$$

which implies

$$I_1 \lesssim \frac{1}{|B|^{1/q}}\left(\int_{2B} |f^q(x)|\, dx\right)^{1/q} \lesssim \frac{1}{|2B|^{1/q}}\left(\int_{2B} |f^q(x)|\, dx\right)^{1/q} \lesssim M_q f(x_0).$$

For I_2, we use the conditions of standard kernels in the definition of Calderón-Zygmund operators of type θ to obtain

$$I_2 = \frac{1}{|B|}\int_B |Tf_2 - (Tf_2)_B|\, dx$$
$$\leqslant \frac{1}{|B|}\int_B \left|\int_{\mathbb{R}^n} K(x,y) f_2(y) dy - \frac{1}{|B|}\int_B\int_{\mathbb{R}^n} K(z,y) f_2(y) dydz\right| dx$$
$$\leqslant \frac{1}{|B|}\int_B \left|\frac{1}{|B|}\int_B\int_{\mathbb{R}^n} K(x,y) f_2(y) dydz - \frac{1}{|B|}\int_B\int_{\mathbb{R}^n} K(z,y) f_2(y) dydz\right| dx$$
$$\leqslant \frac{1}{|B|}\frac{1}{|B|}\int_B \left|\int_B\int_{\mathbb{R}^n} f_2(y)\left(K(x,y) - K(z,y)\right) dydz\right| dx$$
$$\leqslant \frac{1}{|B|}\frac{1}{|B|}\int_B\int_B\int_{\mathbb{R}^n} |f_2(y)|\left|K(x,y) - K(z,y)\right| dydzdx.$$

Take $z, x \in B$ and $y \notin 2B$. Clearly, $2|x - x_0| < |y - x_0|$ and $2|z - x_0| < |y - x_0|$. Then it follows from the regularity condition of the standard kernel of type θ that

$$\int_{\mathbb{R}^n} |f_2(y)|\left|K(x,y)-K(x_0,y)\right|dy = \int_{\mathbb{R}^n\setminus 2B} |f(y)|\left|K(x,y)-K(x_0,y)\right|dy$$
$$\lesssim \sum_{j=1}^{\infty}\int_{2^{j+1}B\setminus 2^jB}\frac{\theta(|x-x_0|/|y-x_0|)}{|x_0-y|^n}|f(y)|dy$$
$$\lesssim \sum_{j=1}^{\infty}\theta(2^{-j})\frac{1}{|2^{j+1}B|}\int_{2^{j+1}B}|f(y)|dy$$
$$\lesssim \sum_{j=1}^{\infty}\theta(2^{-j})\frac{1}{|2^{j+1}B|}\left(\int_{2^{j+1}B}|f^q(y)|dy\right)^{1/q}$$
$$\lesssim \sum_{j=1}^{\infty}\theta(2^{-j})M_q f(x_0) \lesssim M_q f(x_0)\int_0^1\frac{\theta(t)}{t}dt \lesssim M_q f(x_0).$$

By an argument analogous to $|K(z,y)-K(x_0,y)|$, we also obtain

$$\int_{\mathbb{R}^n} |f_2(y)|\left|K(z,y)-K(x_0,y)\right| \lesssim Mf_q(x_0)dy.$$

This leads to

$$I_2 \lesssim \frac{1}{|B|}\frac{1}{|B|}\int_B\int_B\int_{\mathbb{R}^n}|f_2(y)|\left|K(x,y)-K(z,y)\right|dydzdx$$
$$\lesssim \frac{1}{|B|}\frac{1}{|B|}\int_B\int_B\int_{\mathbb{R}^n}|f_2(y)|(|K(x,y)-K(x_0,y)|+|K(z,y)-K(x_0,y)|)dydzdx$$
$$\lesssim \frac{1}{|B|}\frac{1}{|B|}\int_B\int_B Mf_q(x_0)dzdx \lesssim M_q f(x_0).$$

Therefore, we deduce that

$$\left(\frac{1}{|B|}\int_B\left\||Tf|-(Tf_2)_B|dx\right|\right)\lesssim I_1+I_2\lesssim M_q f(x_0),$$

which completes the proof.

Lemma 2.5. *Let $b\in BMO(\mathbb{R}^n)$ and T be a Calderón-Zygmund operator of type θ with $\int_0^1\theta(t)t^{-1}\left|\log t\right|dt<\infty$. Then, for any $1<q<p$, there exists a positive constant C such that*

$$M^{\#}\left([b,T](f)\right)(x_0)\le C\|b\|_{BMO}\left(M_q\left(T(f)\right)(x_0)+M_q f(x_0)\right),$$

for any $f\in\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$ and $x_0\in\mathbb{R}^n$.

Proof. It is sufficient to prove that for any $1<q<p$, any ball $B=B(x_0,r)$, $f\in\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$ and for some constant c, there exists $C=C_q>0$ such that

$$\left(\frac{1}{|B|}\int_B\left\|[b,T]f(y)\right|-|c|\|dy\right)\leqslant C\|b\|_{BMO}\left(M_q\left(T(f)\right)(x_0)+M_q f(x_0)\right).$$

We write $f=f_1+f_2$, with $f_1=f\chi_{2B}$ and $f_2=f\chi_{\mathbb{R}^n\setminus 2B}$. Then, it is clear to verify that

$$[b,T]f=(b-b_{2B})Tf-T((b-b_{2B})f_1)-T((b-b_{2B})f_2).$$

Now, if we take $c=-(T((b-b_{2B})f_2))_B$ then

$$\frac{1}{|B|}\int_B\left||[b,T]f(y)|-|-(T((b-b_{2B})f_2))_B|\right|dy\le\frac{1}{|B|}\int_B\left|[b,T]f(y)+(T((b-b_{2B})f_2))_B\right|dy$$

$$=\frac{1}{|B|}\int_B\left((b(y)-b_{2B})Tf(y)-T\left((b-b_{2B})f_1\right)(y)-T\left((b-b_{2B})f_2\right)(y)+\left(T\left((b-b_{2B})f_2\right)\right)_B\right)dy$$

$$\lesssim\frac{1}{|B|}\int_B|(b(y)-b_{2B})||Tf(y)|dy+\frac{1}{|B|}\int_B\left|T\left((b-b_{2B})f_1\right)(y)\right|dy$$

$$+\frac{1}{|B|}\int_B\left|T\left((b-b_{2B})f_2\right)(y)-\left(T\left((b-b_{2B})f_2\right)\right)_B\right|dy.$$

Set $I_1=\frac{1}{|B|}\int_B|(b(y)-b_{2B})||Tf(y)|dy$, $I_2=\frac{1}{|B|}\int_B\left|T\left((b-b_{2B})f_1\right)(y)\right|dy$ and

$$I_3=\frac{1}{|B|}\int_B\left|T\left((b-b_{2B})f_2\right)(y)-\left(T\left((b-b_{2B})f_2\right)\right)_B\right|dy.$$

For I_1, applying Holder's inequality with exponents q and q' gives

$$I_1\lesssim\left(\frac{1}{|B|}\int_B|(b(y)-b_{2B})|^{q'}dy\right)^{1/q'}\left(\frac{1}{|B|}|Tf(y)|^q\,dy\right)^{1/q}\lesssim\|b\|_{BMO}M_q\left(T(f)\right)(x_0).$$

For I_2, let $v\in(1,q)$. Since T is of weak type (v,v), so based on the Kolmogorov's inequality, we get

$$I_2\le\frac{1}{|B|^{1/v}}\left(\int_{\mathbb{R}^n}|(b-b_{2B})f_1(y)|^v dy\right)^{1/v}\lesssim\frac{1}{|2B|^{1/v}}\left(\int_{2B}|(b-b_{2B})f(y)|^v dy\right)^{1/v}$$

$$\lesssim\left(\frac{1}{|2B|}\int_{2B}\left|\left(b(y)-b_{2B}\right)\right|^{\frac{qv}{q-v}}dy\right)^{\frac{q-v}{qv}}\left(\frac{1}{|2B|}\int_{2B}|f(y)|^q dy\right)^{1/q}\lesssim\|b\|_{BMO}M_q(f)(x_0).$$

For I_3, observe that

$$I_3=\frac{1}{|B|}\int_B\left|T\left((b-b_{2B})f_2\right)(y)-\left(T\left((b-b_{2B})f_2\right)\right)_B\right|dy$$

$$\lesssim\frac{1}{|B|^2}\int_B\int_B\int_{\mathbb{R}^n\backslash 2B}|K(y,w)-K(z,w)||(b(w)-b_{2B})f(w)|dwdzdy$$

$$\lesssim\frac{1}{|B|^2}\int_B\int_B\sum_{j=1}^{\infty}\int_{2^{j+1}B\backslash 2^jB}\frac{\theta(|y-z|/|x_0-w|)}{|x_0-w|^n}|b(w)-b_{2B}||f(w)|dwdzdy$$

$$\lesssim \sum_{j=1}^{\infty} \theta\left(2^{-j}\right) \frac{1}{\left|2^{j+1} B\right|} \int_{2^{j+1} B} \left|b(w)-b_{2B}\right|\left|f(w)\right| dw$$

$$\lesssim \sum_{j=1}^{\infty} \theta\left(2^{-j}\right) \frac{1}{\left|2^{j+1} B\right|} \int_{2^{j+1} B} \left|b(w)-b_{2^{j+1}B}\right|\left|f(w)\right| dw +$$

$$\sum_{j=1}^{\infty} \theta\left(2^{-j}\right) \frac{1}{\left|2^{j+1} B\right|} \int_{2^{j+1} B} \left|b_{2^{j+1}B}-b_{2B}\right|\left|f(w)\right| dw.$$

Let $I_{3a}=\sum_{j=1}^{\infty} \theta\left(2^{-j}\right) \frac{1}{\left|2^{j+1} B\right|} \int_{2^{j+1} B} \left|b(w)-b_{2^{j+1}B}\right|\left|f(w)\right| dw$ and

$$I_{3b}=\sum_{j=1}^{\infty} \theta\left(2^{-j}\right) \frac{1}{\left|2^{j+1} B\right|} \int_{2^{j+1} B} \left|b_{2^{j+1}B}-b_{2B}\right|\left|f(w)\right| dw.$$

For I_{3a}, applying Holder's inequality with exponents q and q' yields

$$I_{3a} \lesssim \sum_{j=1}^{\infty} \theta\left(2^{-j}\right)\left(\frac{1}{\left|2^{j+1} B\right|} \int_{2^{j+1} B}\left|b(w)-b_{2^{j+1}B}\right|^{q'} dw\right)^{1/q'}\left(\frac{1}{\left|2^{j+1} B\right|} \int_{2^{j+1} B}\left|f^{q}(w)\right| dw\right)^{1/q}$$

$$\lesssim \|b\|_{BMO} M_q(f)(x_0) \int_0^1 \frac{\theta(t)}{t} dt \lesssim \|b\|_{BMO} M_q(f)(x_0).$$

For I_{3b}, in view of Lemma 2.3 for each pair of balls $2^{k+1}B$ and $2^k B$ with $k=1,2,\ldots,j$ together with Holder's inequality with exponents q and q', we derive

$$I_{3b} \lesssim \sum_{j=1}^{\infty} \theta\left(2^{-j}\right) j\|b\|_{BMO} \frac{1}{\left|2^{j+1} B\right|} \int_{2^{j+1} B}\left|f(w)\right| dw$$

$$\lesssim \sum_{j=1}^{\infty} \theta\left(2^{-j}\right) j\|b\|_{BMO}\left(\frac{1}{\left|2^{j+1} B\right|} \int_{2^{j+1} B}\left|f^{q}(w)\right| dw\right)^{1/q}$$

$$\lesssim \|b\|_{BMO} M_q(f)(x_0) \int_0^1 \frac{\theta(t)}{t}|\log t| dt \lesssim \|b\|_{BMO} M_q(f)(x_0).$$

Therefore, $I_3 \lesssim \|b\|_{BMO} M_q(f)(x_0)$.

Finally, we deduce from the above estimates that

$$\left(\frac{1}{|B|}\int_B \left\|[b,T] f(y)\right|-|c|\right| dy\right) \lesssim I_1+I_2+I_3 \lesssim \|b\|_{BMO}\left(M_q(T(f))(x_0)+M_q f(x_0)\right).$$

2.2. Main Results

The following section proves the following main results of this study:

Theorem 2.1. *Let* $p \in (1,\infty)$, $r \in [1,\infty)$, *and let* φ *satisfy (1.1). Let* T *be a Calderón-Zygmund operator of type* θ. *Then,* T *maps* $\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n) \to \mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$. *Moreover, we have*

$$\|T(f)\|_{\mathbf{M}_{\varphi}^{p,r}} \lesssim \|f\|_{\mathbf{M}_{\varphi}^{p,r}},$$

for any $f \in \mathbf{M}_{\varphi}^{p,r}\left(\mathbb{R}^n\right)$.

Proof. For any $f \in \mathbf{M}_{\varphi}^{p,r}\left(\mathbb{R}^n\right)$, we take some $q \in (1, p)$. Then in light of Lemma 2.2, it is clear to see that

$$\|M_q(f)\|_{\mathbf{M}_{\varphi}^{p,r}} \lesssim \|f\|_{\mathbf{M}_{\varphi}^{p,r}}.$$

In addition, it follows from Lemma 2.4 that

$$\|M^{\#}(Tf)\|_{\mathbf{M}_{\varphi}^{p,r}} \lesssim \|M_q f\|_{\mathbf{M}_{\varphi}^{p,r}}.$$

Therefore, we deduce that

$$\|Tf\|_{\mathbf{M}_{\varphi}^{p,r}} \lesssim \|M^{\#}(Tf)\|_{\mathbf{M}_{\varphi}^{p,r}} \lesssim \|M_q f\|_{\mathbf{M}_{\varphi}^{p,r}} \lesssim \|f\|_{\mathbf{M}_{\varphi}^{p,r}},$$

which completes the proof of Theorem 2.1.

Theorem 2.2. *Let* $p \in (1, \infty)$, $r \in [1, \infty)$, *and let* φ *satisfy (1.1). Let* T *be a Calderón-Zygmund operator of type* θ *with* $\int_0^1 \theta(t) t^{-1} |\log t| dt < \infty$. *If* $b \in BMO\left(\mathbb{R}^n\right)$ *then* $[b, T]$ *maps* $\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n) \to \mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$. *Moreover, we have*

$$\|[b,T](f)\|_{\mathbf{M}_{\varphi}^{p,r}} \lesssim \|b\|_{BMO} \|f\|_{\mathbf{M}_{\varphi}^{p,r}},$$

for any $f \in \mathbf{M}_{\varphi}^{p,r}\left(\mathbb{R}^n\right)$.

Proof. For any $f \in \mathbf{M}_{\varphi}^{p,r}\left(\mathbb{R}^n\right)$, we take some $q \in (1, p)$. Then in view of Lemmas 2.2, 2.5 and Theorem 2.1, we deduce that

$$\|[b,T](f)\|_{\mathbf{M}_{\varphi}^{p,r}} \lesssim \left\|M^{\#}\left([b,T](f)\right)\right\|_{\mathbf{M}_{\varphi}^{p,r}} \lesssim \|b\|_{BMO} \left\|M_q\left(T(f)\right) + M_q(f)\right\|_{\mathbf{M}_{\varphi}^{p,r}}$$

$$\lesssim \|b\|_{BMO} \left(\left\|M_q\left(T(f)\right)\right\|_{\mathbf{M}_{\varphi}^{p,r}} + \left\|M_q(f)\right\|_{\mathbf{M}_{\varphi}^{p,r}} \right)$$

$$\lesssim \|b\|_{BMO} \left(\|T(f)\|_{\mathbf{M}_{\varphi}^{p,r}} + \|f\|_{\mathbf{M}_{\varphi}^{p,r}} \right) \lesssim \|b\|_{BMO} \|f\|_{\mathbf{M}_{\varphi}^{p,r}}.$$

3. Conclusion

By employing the ideas and techniques of Liu et al. (2001) and Thai et al. (2022a, 2022b), we obtain the boundedness of Calderón-Zygmund operators T of type θ and its commutators on the generalized Morrey – Lorentz space $\mathbf{M}_{\varphi}^{p,r}\left(\mathbb{R}^n\right)$ (Theorems 2.1 and 2.2).

Result 1. *Let* $p \in (1,\infty)$, $r \in [1,\infty)$, *and let* $\varphi(t)$ *satisfy (1.1). Let* T *be a* θ *type Calderón-Let* $p \in (1,\infty)$, $r \in [1,\infty)$, *and let* φ *satisfy (1.1). Let* T *be a Calderón-Zygmund operator of type* θ. *Then,* T *maps* $\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n) \to \mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$. *Moreover, we have*

$$\|T(f)\|_{\mathbf{M}_{\varphi}^{p,r}} \lesssim \|f\|_{\mathbf{M}_{\varphi}^{p,r}},$$

for any $f \in \mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$.

Result 2. *Let* $p \in (1,\infty)$, $r \in [1,\infty)$, *and let* φ *satisfy (1.1). Let* T *be a Calderón-Zygmund operator of type* θ *with* $\int_0^1 \theta(t)t^{-1}|\log t|\,dt < \infty$. *If* $b \in BMO(\mathbb{R}^n)$ *then* $[b,T]$ *maps* $\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n) \to \mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$. *Moreover, we have*

$$\|[b,T](f)\|_{\mathbf{M}_{\varphi}^{p,r}} \lesssim \|b\|_{BMO}\|f\|_{\mathbf{M}_{\varphi}^{p,r}},$$

for any $f \in \mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$.

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HOÁN TỬ CALDERÓN-ZYGMUND LOẠI THETA TRÊN KHÔNG GIAN MORREY-LORENTZ TỔNG QUÁT

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TÓM TẮT

Trong bài báo này, chúng tôi xét hoán tử Calderón-Zygmund loại θ (xem Định nghĩa 1.3, 1.4 và 1.5 trong Phần 1) trong không gian Morrey – Lorentz tổng quát $\mathbf{M}_{\varphi}^{p,r}\left(\mathbb{R}^n\right)$ (xem Định nghĩa 1.1). Trước tiên chúng tôi thiết lập đánh giá điểm cho toán tử cực đại Hardy-Littlewood và toán tử cực đại chặt tác động lên toán tử Calderón-Zygmund loại θ và hoán tử của nó (xem Bổ đề 2.4 và 2.5 trong Phần 2) bằng cách sử dụng bất đẳng thức Kolmogorov, bất đẳng thức Holder, các điều kiện của nhân chuẩn trong định nghĩa của toán tử Calderón-Zygmund loại θ và hệ quả nổi tiếng của bất đẳng thức John-Nirenberg. Sử dụng các đánh giá điểm quan trọng này, chúng tôi chứng minh được rằng các toán tử Calderón-Zygmund loại θ bị chặn trên không gian Morrey – Lorentz tổng quát $\mathbf{M}_{\varphi}^{p,r}\left(\mathbb{R}^n\right)$ (xem Định lí 2.1) dựa theo ý tưởng và kĩ thuật liên quan đến toán tử cực đại trong công trình của Thai et al. (2022), Carro et al. (2021) và Liu et al. (2002). Hơn nữa, kết hợp đánh giá điểm cho toán tử cực đại nhọn tác động lên hoán tử Calderón-Zygmund loại θ và tính bị chặn của toán tử Calderón-Zygmund trên $\mathbf{M}_{\varphi}^{p,r}\left(\mathbb{R}^n\right)$, chúng tôi chứng minh được hoán tử $[b,T]$ loại θ cũng bị chặn trên không gian này.

Từ khóa: hoán tử Calderón-Zygmund loại θ; không gian Morrey – Lorentz tổng quát; toán tử cực đại