TẠP CHÍ KHOA HỌC
TRƯỜNG ĐẠI HỌC SƯ PHẠM TP HỒ CHÍ MINH
Tập 22, Số 3 (2025): 424-436
Website: https://journal.hcmue.edu.vn
ISSN: 2734-9918

HO CHI MINH CITY UNIVERSITY OF EDUCATION
JOURNAL OF SCIENCE
Vol. 22, No. 3 (2025): 424-436
https://doi.org/10.54607/hcmue.js.22.3.3999(2025)

Research Article

A GENERALIZED DISTRIBUTIONAL INEQUALITY AND APPLICATIONS

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Received: October 13, 2023; Revised: March 30, 2024; Accepted: June 05, 2024

ABSTRACT

The distributional inequality recently introduced by Tran and Nguyen has been used to investigate gradient estimates for solutions to partial differential equations. In particular, the authors established several sufficient conditions under which two measurable functions can be compared via their norms in general Lebesgue spaces. The results are then applied to some classes of p-Laplace type problems. This paper extends this inequality to make it applicable to a broader range of equations. Specifically, we propose a generalized distributional inequality that can be applied to the p(x)-Laplace equation, the typical version of quasi-linear elliptic equations with variable exponents.

Keywords: Generalized distributional inequality; Lorentz spaces; *p(x)*-Laplace equation; Quasi-linear elliptic problems; Regularity theory; Variable exponents

1. Motivation and introduction

Let Ω be an open bounded domain in $\mathbb{R}^n$ and $\mathcal{F}$, $\mathcal{G}$ be two Lebesgue measurable functions defined in Ω. In recent papers, Nguyen et al. (2021) and Nguyen and Tran (2021) proved the following distribution inequality.

$$d_{\mathcal{G}}^{\alpha}\left(\varepsilon^{-a}\lambda\right) \leq C\varepsilon d_{\mathcal{G}}^{\alpha}\left(\lambda\right) + d_{\mathcal{F}}^{\alpha}\left(\varepsilon^{b}\lambda\right), \tag{1.1}$$

for all $\lambda > 0$ and $\varepsilon > 0$ small enough, under some sufficient conditions of $\mathcal{F}$ and $\mathcal{G}$. Here, a and b are two positive constants, and the distribution function d_h^{α} is considered as the Lebesgue measure of level sets corresponding to the fractional maximal operators $\mathbf{M}_{\alpha}$. More precisely, d_h^{α} is defined by $d_h^{\alpha}(\lambda) := \left|\left\{x \in \Omega : \mathbf{M}_{\alpha} h(x) > \lambda\right\}\right|$ for $\lambda > 0$, where h is measurable in Ω and $\mathbf{M}_{\alpha}$ is the fractional maximal operator (see Definition 2.2). The most interesting point is that the distribution inequality (1.1) implies the following statement

$$\mathbf{M}_{\alpha}\mathcal{F} \in \mathbb{X} \Rightarrow \mathbf{M}_{\alpha}\mathcal{G} \in \mathbb{Y}, \tag{1.2}$$

Cite this article as: Le, K. H. (2025). A generalized distributional inequality and applications. *Ho Chi Minh City University of Education Journal of Science, 22*(3), 424-436. https://doi.org/10.54607/hcmue.js.22.3.3999(2025)

in several function spaces $\mathbb{X}$ and $\mathbb{Y}$, such as Lebesgue space, Lorentz space, or Orlicz space. The statement in (1.2) has many meanings about the fractional maximal operator, which is closely related to the fractional order derivative and Riesz potential (Muckenhoupt & Wheeden, 1974). On the other hand, using the boundedness property of maximal operators $\mathbf{M}$, one can obtain that $\mathcal{F} \in \mathbb{X} \Rightarrow \mathcal{G} \in \mathbb{Y}$.

If we present $\mathcal{F}$ and $\mathcal{G}$ as the terms of data and solutions of partial differential equations, respectively, then we can obtain a regularity result for solutions. This type of result has been studied in many works (Acerbi & Mingione, 2005, 2007; Byun & Ok, 2016). By (1.2), Nguyen and Tran (2021) presented some applications to obtain the regularity results for some classes of quasi-linear elliptic equations, such as the *p*-Laplace equation and the obstacle problems associated with the *p*-Laplace operator. The method can be applied to many other problems (Nguyen et al., 2023; Tran & Nguyen, 2022a, 2022b, 2023; Tran et al., 2023). However, it cannot be applied to several cases of problems, such as the quasi-linear elliptic systems, the non-linear problems with variable exponents, and the measure data problems. Motivated by these works, this study aims to establish a new distribution inequality that is more general than (1.1) for applying to larger classes of partial differential equations.

We now introduce some general notations. We write $B_r(x)$ to indicate the ball with radius $r > 0$ and centered at $x \in \mathbb{R}^n$ and $\Omega_r(x) = B_r(x) \cap \Omega$. Next, notation $|B|$ presents the Lebesgue measure of the measurable subset $B \subset \mathbb{R}^n$. For simplicity of notation, the set $\{|h| > \lambda\}$ will be used instead of $\{x \in \Omega : |h(x)| > \lambda\}$. On the other hand, we always consider the letter C as a general constant, and its value may change from different lines of estimates. Let us consider two given vector-valued functions u and $v : \Omega \to \mathbb{R}^m$ with $m \geq 1$. Assume that f, $g : \mathbb{R}^m \to [0, \infty)$ and $F : \Omega \to [0, \infty)$ satisfy all the following assumptions. We will write $f(u)$ instead of $f(u(z))$ with $z \in \Omega$.

Assumption 1.1. There exists a constant $C > 0$ not depending on u and v such that

$$f(u) \leq C\left[f(u-v) + f(v)\right] \quad \text{and} \quad f(v) \leq C\left[f(u-v) + f(u)\right], \quad \text{in } \Omega.$$

Assumption 1.2. There exists a constant $C > 0$ not depending on f, g and F such that

$$\int_{\Omega} f(u)\mathrm{dz} \leq C\int_{\Omega}\left[g(u) + F(z)\right]\mathrm{dz}.$$

Assumption 1.3. There exists a constant $R_0 > 0$ and $\Theta > 1$ satisfying

$$\left(\frac{1}{|\Omega_r(x_0)|}\int_{\Omega_r(x_0)}\left[f(v)\right]^{\Theta}\mathrm{dz}\right)^{\frac{1}{\Theta}} \leq \frac{C}{|\Omega_{2r}(x_0)|}\int_{\Omega_{2r}(x_0)}\left[1 + f(v)\right]\mathrm{dz} \text{ for all } x_0 \in \overline{\Omega} \text{ and } r \in (0; R_0].$$

Assumption 1.4. There exists a constant $\kappa > 0$ and $C > 0$ such that the following inequality

$$\int_{\Omega_r(x_0)} f(u-v)\mathrm{dz} \leq \tau_1\int_{\Omega_r(x_0)} f(u)\mathrm{dz} + \tau_2\int_{\Omega_r(x_0)} g(u)\mathrm{dz} + C(\tau_1 + \tau_2)^{-\kappa}\int_{\Omega_r(x_0)} F(z)\mathrm{dz},$$

holds for every τ_1, $\tau_2 \in (0,1)$, $r > 0$ and $x_0 \in \overline{\Omega}$. Let us now state our main results under Assumptions 1.1–1.4. The results are studied in two steps. In the first step, we construct the following distribution function inequality

$$d^{\alpha}_{f(u)}\left(\varepsilon^{-a}\lambda\right) \le C\varepsilon d^{\alpha}_{f(u)}(\lambda) + d^{\alpha}_{F}\left(\varepsilon^{b}\lambda\right) + d^{\alpha}_{g(u)}\left(\varepsilon^{c}\lambda\right),$$

for ε small enough and λ large enough. Using this inequality and the definition of norm in Lorentz space $L^{q,s}(\Omega)$, we can show that

$$\|\mathbf{M}_{\alpha}P\|_{L^{q,s}(\Omega)} \le C\left(1 + \|\mathbf{M}_{\alpha}F\|_{L^{q,s}(\Omega)} + \|\mathbf{M}_{\alpha}Q\|_{L^{q,s}(\Omega)}\right)$$

for a suitable value of α, where $P = f(u)$ and $Q = g(u)$.

In the next section, we recall some well-known definitions of Lorentz spaces and fractional maximal operators, and we present key results concerning the boundedness of the maximal operator. In Section 3, we establish the general form of the level set inequality, from which the distributional inequality is derived. Then we also obtain the norm estimates in Lorentz space. These proofs are presented in Section 4. In the last section, we discuss some applications to quasi-linear elliptic problems involving variable exponents.

2. Lorentz spaces and maximal operators

In this section, let us recall the definitions of Lorentz spaces, fractional maximal operators, and distribution functions associated with maximal operators. Moreover, we present some useful boundedness properties of maximal operators.

Definition 2.1. **(Lorentz spaces)** *For some $q \in (0,\infty)$ and $s \in (0,\infty]$, the Lorentz space $L^{q,s}(\Omega)$ is defined as the set of all Lebesgue measurable functions h on Ω such that*

$$\|h\|_{L^{q,s}(\Omega)} := \left[q\int_0^{\infty}\lambda^{s}\left|\left\{x\in\Omega : |h(x)| > \lambda\right\}\right|^{\frac{s}{q}}\frac{\mathrm{d}\lambda}{\lambda}\right]^{\frac{1}{s}} < \infty, \tag{2.1}$$

as $s \ne \infty$ and $\|h\|_{L^{q,\infty}(\Omega)} := \sup_{\lambda>0}\lambda\left|\left\{x\in\Omega : |h(x)| > \lambda\right\}\right|^{\frac{1}{q}} < \infty.$

Definition 2.2. *For every $\alpha \in [0,n]$, the fractional maximal operator $\mathbf{M}_{\alpha}$ is defined by*

$$\mathbf{M}_{\alpha}h(x) = \sup_{r>0}\frac{r^{\alpha}}{|B_r(x)|}\int_{B_r(x)}|h(y)|\,\mathrm{d}y, \quad \text{for } x\in\mathbb{R}^n,\ h\in L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

If $\alpha = 0$, it coincides with the Hardy-Littlewood operator, i.e. $\mathbf{M} = \mathbf{M}_0$. The two following results (Propositions 2.3 and 2.4) can be found in Grafakos (2004).

Proposition 2.3. *The Hardy-Littlewood operator $\mathbf{M}$ is bounded from $L^q(\mathbb{R}^n)$ to $L^{q,\infty}(\mathbb{R}^n)$ for all $q \ge 1$. This means there exists a constant $C > 0$ such that*

$$\left|\left\{x\in\mathbb{R}^n : |h(x)| > \lambda\right\}\right| \le \frac{C}{\lambda^q}\int_{\mathbb{R}^n}|h(x)|^q\,\mathrm{d}x, \ \text{for all } \lambda > 0 \text{ and } h\in L^q(\mathbb{R}^n).$$

Proposition 2.4. *Operator* $\mathbf{M}$ *is bounded in Lorentz space* $L^{q,s}(\mathbb{R}^n)$ *for all* $q>1$ *and* $0<s\le\infty$. *This means there exists a constant* $C>0$ *such that*

$$\|\mathbf{M}h\|_{L^{q,s}(\mathbb{R}^n)} \le C\|h\|_{L^{q,s}(\mathbb{R}^n)}, \quad \text{for all } h\in L^{q,s}(\mathbb{R}^n).$$

Let us now state the boundedness properties of fractional maximal operators (see Tran & Nguyen, 2020 for the proofs).

Proposition 2.5. (Tran & Nguyen, 2020) *For all* $\alpha\in[0,n)$, *there holds*

$$\left|\left\{x\in\mathbb{R}^n : \mathbf{M}_\alpha h(x)>\lambda\right\}\right| \le C\left(\frac{1}{\lambda}\int_{\mathbb{R}^n}|h(y)|\,\mathrm{d}y\right)^{\frac{n}{n-\alpha}}, \quad \text{for all } \lambda>0 \text{ and } h\in L^1(\mathbb{R}^n).$$

Corollary 2.6. *Let* $0\le\alpha<n$, $r>0$ *and* $x\in\mathbb{R}^n$. *Then exists a constant* $C>0$ *satisfying*

$$\left|\left\{\mathbf{M}_\alpha\left(\chi_{B_r(x)}h\right)>\lambda\right\}\right| \le C\left(\frac{1}{\lambda}\int_{B_r(x)}|h(y)|\,\mathrm{d}y\right)^{\frac{n}{n-\alpha}}, \quad \text{for all } \lambda>0 \text{ and } h\in L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

Proposition 2.7. (Tran & Nguyen, 2020) *Assume* $s\ge1$ *and* $\alpha\in\left[0,\frac{n}{s}\right)$. *Then exists a positive constant* $C=C(n,s,\alpha)>0$ *such that*

$$\left|\left\{\mathbf{M}_\alpha h>\lambda\right\}\right| \le C\left(\frac{1}{\lambda^s}\int_{\mathbb{R}^n}|h(y)|^s\,\mathrm{d}y\right)^{\frac{n}{n-\alpha s}}, \quad \text{for all } \lambda>0 \text{ and } h\in L^s(\mathbb{R}^n).$$

Definition 2.8. *Let* $\alpha\in[0,n)$ *and* h *be a measurable function in* Ω. *We define by* d_h^α *the distribution function of* h *as* $d_h^\alpha(\lambda):=\left|\left\{x\in\Omega : \mathbf{M}_\alpha h(x)>\lambda\right\}\right|, \quad \lambda\ge0.$

3. Main results

3.1. Generalized level-set inequality

We first prove a generalized level-set inequality, which can be obtained by the following covering lemma.

Lemma 3.1. (Covering & Peral, 1998) *Let* $R_0>0$ *and* $\mathcal{V}\subset\mathcal{W}$ *be two measurable subsets of* Ω. *Assume that* $\varepsilon\in(0,1)$ *satisfying* $|\mathcal{V}|\le\varepsilon|B_{R_0}(0)|$. *Moreover,* $x\in\Omega$ *and* $r\in(0,R_0]$, $|\mathcal{V}\cap B_r(x)|>\varepsilon|B_r(x)| \Rightarrow (\Omega\cap B_r(x))\subset\mathcal{W}$. *Then, there exists a constant* $C=C(n)>0$ *such that* $|\mathcal{V}|\le C\varepsilon|\mathcal{W}|$.

From now on, for simplicity of notation, let us denote $P=f(u)$ and $Q=g(u)$.

Theorem 3.2. *We assume that all Assumptions 1.1–1.4 are satisfied. For every* $\alpha\in\left[0,\frac{n}{\Theta}\right)$ *and* $a>\frac{n-\Theta\alpha}{n\Theta}$, *there exist positive constants* λ_0, ε_0, b, c *and* $C=C\left(n,\alpha,s,\frac{D}{R_0}\right)>0$

such that the distribution inequality $\left|\mathcal{V}_{\varepsilon,\lambda}\right| \le C\varepsilon\left|\mathcal{W}_{\lambda}\right|$ *holds for all* $\varepsilon \in (0,\varepsilon_0)$ *and* $\lambda > \lambda_0$. *Here* $\mathcal{V}_{\varepsilon,\lambda}$, $\mathcal{W}_{\lambda}$ *are defined by*

$$\mathcal{V}_{\varepsilon,\lambda} = \left\{\mathbf{M}_\alpha P > \varepsilon^{-a}\lambda; \mathbf{M}_\alpha F \le \varepsilon^{b}\lambda; \mathbf{M}_\alpha Q \le \varepsilon^{c}\lambda\right\} \quad \textit{and} \quad \mathcal{W}_{\lambda} = \left\{\mathbf{M}_\alpha P > \lambda\right\}.$$

Proof. The proof will be divided into two steps. In the first step, let us prove that for all $\varepsilon > 0$ small enough and $\lambda > \lambda_0$, there holds

$$\left|\mathcal{V}_{\varepsilon,\lambda}\right| \le \varepsilon\left|B_{R_0}(0)\right|. \tag{3.1}$$

We assume $\mathcal{V}_{\varepsilon,\lambda} \ne \varnothing$, since (3.1) is valid if $\mathcal{V}_{\varepsilon,\lambda} = \varnothing$. Then there exists $x_1 \in \Omega$ such that

$$\begin{cases} \mathbf{M}_\alpha F(x_1) \le \varepsilon^{b}\lambda \Leftrightarrow \forall \eta > 0 : \dfrac{\eta^\alpha}{\left|B_\eta(x_1)\right|}\displaystyle\int_{B_\eta(x_1)} F(x)\,\mathrm{d}x \le \varepsilon^{b}\lambda, \\ \mathbf{M}_\alpha Q(x_1) \le \varepsilon^{c}\lambda \Leftrightarrow \forall \eta > 0 : \dfrac{\eta^\alpha}{\left|B_\eta(x_1)\right|}\displaystyle\int_{B_\eta(x_1)} Q(x)\,\mathrm{d}x \le \varepsilon^{c}\lambda. \end{cases} \tag{3.2}$$

Let us set $D = 2\mathrm{diam}(\Omega)$ and $\mathrm{B} = B_D(x_1)$, where $\mathrm{diam}(\Omega)$ denotes the diameter of Ω. By Proposition 2.7 for $s = 1$, we can see that

$$\left|\mathcal{V}_{\varepsilon,\lambda}\right| \le \left|\left\{\mathbf{M}_\alpha P > \varepsilon^{-a}\lambda\right\}\right| \le \left(\frac{C}{\varepsilon^{-a}\lambda}\int_\Omega P(x)\,\mathrm{d}x\right)^{\frac{n}{n-\alpha}}. \tag{3.3}$$

Using Assumption 1.2, we observe that

$$\int_\Omega P(x)\,\mathrm{d}x \le C\int_\Omega \left[Q(x) + F(x)\right]\mathrm{d}x. \tag{3.4}$$

Substituting (3.4) into (3.3), we can rewrite (3.3) as

$$\left|\mathcal{V}_{\varepsilon,\lambda}\right| \le \left(\frac{C}{\varepsilon^{-a}\lambda}\int_\Omega \left[Q(x) + F(x)\right]\mathrm{d}x\right)^{\frac{n}{n-\alpha}} \le \left(\frac{C}{\varepsilon^{-a}\lambda}\left(\int_{\mathrm{B}} Q(x)\,\mathrm{d}x + \int_{\mathrm{B}} F(x)\,\mathrm{d}x\right)\right)^{\frac{n}{n-\alpha}}. \tag{3.5}$$

From (3.2), and choosing $\eta = D = 2\mathrm{diam}(\Omega)$, we have

$$\frac{D^\alpha}{|\mathrm{B}|}\int_{\mathrm{B}} F(x)\,\mathrm{d}x \le \varepsilon^{b}\lambda, \text{ and } \frac{D^\alpha}{|\mathrm{B}|}\int_{\mathrm{B}} Q(x)\,\mathrm{d}x \le \varepsilon^{c}\lambda.$$

It follows that

$$\int_{\mathrm{B}} F(x)\,\mathrm{d}x \le C(n)D^{n-\alpha}\varepsilon^{b}\lambda, \text{ and } \int_{\mathrm{B}} Q(x)\,\mathrm{d}x \le C(n)D^{n-\alpha}\varepsilon^{c}\lambda. \tag{3.6}$$

Replacing (3.5) by (3.6), we obtain

$$\left|\mathcal{V}_{\varepsilon,\lambda}\right| \le \left(\frac{C}{\varepsilon^{-a}\lambda}\left[C(n)D^{n-\alpha}\left(\varepsilon^{b} + \varepsilon^{c}\right)\lambda\right]\right)^{\frac{n}{n-\alpha}} \le C\left(\varepsilon^{\frac{(a+b)n}{n-\alpha}-1} + \varepsilon^{\frac{(a+c)n}{n-\alpha}-1}\right)\varepsilon\left(\frac{D}{R_0}\right)^n\left|B_{R_0}(0)\right|.$$

Let us choose positive constants b, c and ε_1 satisfying conditions

$$\frac{(a+b)n}{n-\alpha}-1>0,\ \frac{(a+c)n}{n-\alpha}-1>0,\ C\left(\frac{D}{R_0}\right)^n\left(\varepsilon_1^{\frac{(a+b)n}{n-\alpha}-1}+\varepsilon_1^{\frac{(a+c)n}{n-\alpha}-1}\right)<1. \tag{3.7}$$

It allows us to conclude (3.1) for every $\varepsilon\in(0,\varepsilon_1)$.

Let us consider the second step of the proof. For all $x\in\Omega$ and $r\in(0,R]$, assume that $\Omega\cap B_r(x)\not\subset\mathcal{W}_\lambda$, we need to prove that

$$\left|\mathcal{V}_{\varepsilon,\lambda}\cap B_r(x)\right|\le\varepsilon\left|B_r(x)\right|. \tag{3.8}$$

Inequality (3.8) holds if $\mathcal{V}_{\varepsilon,\lambda}\cap B_r(x)=\varnothing$, so let us suppose $\mathcal{V}_{\varepsilon,\lambda}\cap B_r(x)\neq\varnothing$. Then there exist $x_2\in\Omega\cap B_r(x)\cap\mathcal{W}_\lambda^c$ and $x_3\in\mathcal{V}_{\varepsilon,\lambda}\cap B_r(x)$, which means

$$\mathbf{M}_\alpha P(x_2)\le\lambda,\ \mathbf{M}_\alpha F(x_3)\le\varepsilon^b\lambda,\ \text{and } \mathbf{M}_\alpha Q(x_3)\le\varepsilon^c\lambda. \tag{3.9}$$

We now prove that the operator $\mathbf{M}_\alpha P$ can be replaced by the cut-off operator $\mathbf{M}_\alpha^r P$ when ε is small enough. Indeed, for all $y\in\Omega$, it is easy to check that

$$\mathbf{M}_\alpha P(y)=\max\left\{\mathbf{M}_\alpha^r P(y);\mathbf{T}_\alpha^r P(y)\right\}, \tag{3.10}$$

where the cut-off operators $\mathbf{M}_\alpha^r$ and $\mathbf{T}_\alpha^r$ are defined by

$$\mathbf{M}_\alpha^r P(y)=\sup_{0<\eta<r}\frac{\eta^\alpha}{\left|B_\eta(y)\right|}\int_{B_\eta(y)}P(z)\mathrm{d}z,\quad \mathbf{T}_\alpha^r P(y)=\sup_{\eta\ge r}\frac{\eta^\alpha}{\left|B_\eta(y)\right|}\int_{B_\eta(y)}P(z)\mathrm{d}z.$$

We have $\left|\mathcal{V}_{\varepsilon,\lambda}\cap B_r(x)\right|\le\left|\left\{y\in B_r(x):\mathbf{M}_\alpha P(y)>\varepsilon^{-a}\lambda\right\}\right|$, which by (3.10) yields that

$$\left|\mathcal{V}_{\varepsilon,\lambda}\cap B_r(x)\right|\le\left|\left\{y\in B_r(x):\mathbf{M}_\alpha^r P(y)>\varepsilon^{-a}\lambda\right\}\right|+\left|\left\{y\in B_r(x):\mathbf{T}_\alpha^r P(y)>\varepsilon^{-a}\lambda\right\}\right|. \tag{3.11}$$

Getting $y\in B_r(x)$, for all $\eta\ge r$ and $z\in B_\eta(y)$, $B_\eta(y)\subset B_{3\eta}(x_2)$. Hence, one has

$$\mathbf{T}_\alpha^r P(y)\le 3^{n-\alpha}\sup_{\eta\ge r}(3\eta)^\alpha\frac{1}{\left|B_{3\eta}(x_2)\right|}\int_{B_{3\eta}(x_2)}P(z)\mathrm{d}z\le 3^{n-\alpha}\mathbf{M}_\alpha P(x_2).$$

Using inequality (3.9), we have $\mathbf{T}_\alpha^r P(y)\le 3^{n-\alpha}\lambda,\quad\forall y\in B_r(x)$. For all $\varepsilon>0$ satisfying

$\varepsilon^{-a}>3^{n-\alpha}\Leftrightarrow\varepsilon<\varepsilon_2:=3^{-\frac{n-\alpha}{a}}$, it is easy to see that

$$\left\{y\in B_r(x):\mathbf{T}_\alpha^r P(y)>\varepsilon^{-a}\lambda\right\}=\varnothing. \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$\left|\mathcal{V}_{\varepsilon,\lambda}\cap B_r(x)\right|\le\left|\left\{y\in B_r(x):\mathbf{M}_\alpha^r P(y)>\varepsilon^{-a}\lambda\right\}\right|. \tag{3.13}$$

Further, for all $y\in B_r(x)$ and $\eta\in(0,r)$, it is clearly to see that $B_\eta(y)\subset B_{2r}(x)$ and

$$\mathbf{M}_\alpha^r P(y)=\sup_{0<\eta<r}\frac{\eta^\alpha}{\left|B_\eta(y)\right|}\int_{B_\eta(y)}\chi_{B_{2r}(x)}P(z)\mathrm{d}z=\mathbf{M}_\alpha^r\left(\chi_{B_{2r}(x)}P\right)(y). \tag{3.14}$$

Substituting (3.14) into (3.13), there holds

$$\left|\mathcal{V}_{\varepsilon,\lambda}\cap B_r(x)\right|\le\left|\left\{\mathbf{M}_\alpha^r\left(\chi_{B_{2r}(x)}P\right)>\varepsilon^{-a}\lambda\right\}\right|. \tag{3.15}$$

for all $\varepsilon\in(0,\varepsilon_2)$. Next, we apply Assumptions 1.3 and 1.4 to prove inequality (3.8). We consider two cases. In the first case, let us assume that $B_{4r}(x)\subset\Omega$. Using Assumption 1.4, we have $C>0$ and $\kappa>0$ such that

$$\int_{B_{4r}(x)} f(u-v)\mathrm{d}z\le\tau_1\int_{B_{4r}(x)} f(u)\mathrm{d}z+\tau_2\int_{B_{4r}(x)} g(u)\mathrm{d}z+C(\tau_1+\tau_2)^{-\kappa}\int_{B_{4r}(x)} F(z)\mathrm{d}z,$$

for all τ_1, $\tau_2\in(0,1)$. Since $x_2\in B_r(x)$, we can check that $B_{4r}(x)\subset B_{5r}(x_2)$. It leads to

$$\begin{aligned}\frac{1}{\left|B_{4r}(x)\right|}\int_{B_{4r}(x)} P(z)\mathrm{d}z&=\frac{\left|B_{5r}(x_2)\right|}{\left|B_{4r}(x)\right|}\cdot\frac{\int_{B_{4r}(x)} P(z)\mathrm{d}z}{\left|B_{5r}(x_2)\right|}\le\left(\frac{5}{4}\right)^n\frac{\int_{B_{5r}(x_2)} P(z)\mathrm{d}z}{\left|B_{5r}(x_2)\right|}\\&\le\left(\frac{5}{4}\right)^n(5r)^{-\alpha}\mathbf{M}_\alpha P(x_2)\le\left(\frac{5}{4}\right)^n(5r)^{-\alpha}\lambda=Cr^{-\alpha}\lambda.\end{aligned} \tag{3.16}$$

Similarly, we have $B_{4r}(x)\subset B_{5r}(x_3)$. Using inequality (3.9), it is easily seen that

$$\frac{1}{\left|B_{4r}(x)\right|}\int_{B_{4r}(x)} Q(z)\mathrm{d}z\le\left(\frac{5}{4}\right)^n(5r)^{-\alpha}\mathbf{M}_\alpha Q(x_3)\le\left(\frac{5}{4}\right)^n(5r)^{-\alpha}\varepsilon^c\lambda=Cr^{-\alpha}\varepsilon^c\lambda. \tag{3.17}$$

Performing the same proof, we also have

$$\frac{1}{\left|B_{4r}(x)\right|}\int_{B_{4r}(x)} F(z)\mathrm{d}z\le Cr^{-\alpha}\varepsilon^b\lambda. \tag{3.18}$$

Combining all estimates in (3.16), (3.17), and (3.18), we obtain that

$$\frac{1}{\left|B_{4r}(x)\right|}\int_{B_{4r}(x)} f(u-v)\mathrm{d}z\le C\left(\underbrace{\tau_1+\tau_2\varepsilon^c+(\tau_1+\tau_2)^{-\kappa}\varepsilon^b}_{\tau=\tau(\tau_1,\tau_2)>0}\right)r^{-\alpha}\lambda=C\tau r^{-\alpha}\lambda. \tag{3.19}$$

Using Assumption 1.3, there exists $\Theta>1$ such that

$$\left(\frac{1}{\left|B_{2r}(x)\right|}\int_{B_{2r}(x)}\left[f(v)\right]^\Theta\mathrm{d}z\right)^{\frac{1}{\Theta}}\le\frac{C}{\left|B_{4r}(x)\right|}\int_{B_{4r}(x)}\left(1+f(v)\right)\mathrm{d}z. \tag{3.20}$$

From Assumption 1.1, we have $f(v)\le C\left[f(u-v)+f(u)\right]$. Using (3.16) and (3.19), we obtain

$$\frac{1}{\left|B_{4r}(x)\right|}\int_{B_{4r}(x)} f(v)\mathrm{d}z\le\frac{C}{\left|B_{4r}(x)\right|}\left(\int_{B_{4r}(x)} f(u-v)\mathrm{d}z+\int_{B_{4r}(x)} f(u)\mathrm{d}z\right)\le Cr^{-\alpha}\lambda(\tau+1). \tag{3.21}$$

Substituting (3.21) into (3.20), we get that

$$\frac{1}{\left|B_{2r}(x)\right|}\int_{B_{2r}(x)}\left[f(v)\right]^\Theta\mathrm{d}z\le C\left[r^{-\alpha}\lambda(\tau+1)+1\right]^\Theta. \tag{3.22}$$

Using Assumption 1.1 again, we get $f(u) \leq C\left[f(u-v)+f(v)\right]$ and

$$\mathbf{M}_\alpha^r\left(\chi_{B_{2r}(x)}P\right) \leq C'\left[\mathbf{M}_\alpha^r\left(\chi_{B_{2r}(x)}f(u-v)\right)+\mathbf{M}_\alpha^r\left(\chi_{B_{2r}(x)}f(v)\right)\right]. \tag{3.23}$$

On the other hand, from (3.15), we obtain that

$$\left|\mathcal{V}_{\varepsilon,\lambda} \cap B_r(x)\right| \leq \left|\left\{\mathbf{M}_\alpha^r\left(\chi_{B_{2r}(x)}f(u-v)\right) > C'\varepsilon^{-a}\lambda\right\}\right| + \left|\left\{\mathbf{M}_\alpha^r\left(\chi_{B_{2r}(x)}f(v)\right) > C'\varepsilon^{-a}\lambda\right\}\right|. \tag{3.24}$$

Applying Proposition 2.7 for $s=1$ and $s=\Theta>1$ respectively, we have

$$\left|\left\{\mathbf{M}_\alpha^r\left(\chi_{B_{2r}(x)}f(u-v)\right) > C'\varepsilon^{-a}\lambda\right\}\right| \leq C\left(\frac{1}{C'\varepsilon^{-a}\lambda}\int_{B_{2r}(x)} f(u-v)\,\mathrm{dz}\right)^{\frac{n}{n-\alpha}}, \tag{3.25}$$

and $$\left|\left\{\mathbf{M}_\alpha^r\left(\chi_{B_{2r}(x)}f(v)\right) > C'\varepsilon^{-a}\lambda\right\}\right| \leq C\left(\frac{1}{\left(C'\varepsilon^{-a}\lambda\right)^{\Theta}}\int_{B_{2r}(x)} \left[f(v)\right]^{\Theta}\mathrm{dz}\right)^{\frac{n}{n-\Theta\alpha}}. \tag{3.26}$$

It is seen that $\left|B_{4r}(x)\right| \sim \left|B_{2r}(x)\right| \sim r^n$, and we get

$$\frac{\int_{B_{2r}(x)} f(u-v)\,\mathrm{dz}}{C'\varepsilon^{-a}\lambda} \leq \frac{Cr^n}{\varepsilon^{-a}\lambda}\cdot\frac{\int_{B_{4r}(x)} f(u-v)\,\mathrm{dz}}{\left|B_{4r}(x)\right|} \leq \frac{C\tau r^{n-\alpha}}{\varepsilon^{-a}} = C\tau\varepsilon^a r^{n-\alpha},$$

which will be substituted into (3.25). On the other hand, we have

$$\begin{aligned}\frac{\int_{B_{2r}(x)}\left[f(v)\right]^{\Theta}\mathrm{dz}}{\left(C'\varepsilon^{-a}\lambda\right)^{\Theta}} &\leq \frac{Cr^n}{\left(\varepsilon^{-a}\lambda\right)^{\Theta}}\cdot\frac{\int_{B_{2r}(x)}\left[f(v)\right]^{\Theta}\mathrm{dz}}{\left|B_{2r}(x)\right|} \leq \frac{Cr^n\varepsilon^{\Theta a}}{\lambda^{\Theta}}\cdot\left[r^{-\alpha}\lambda(\tau+1)+1\right]^{\Theta}\\ &\leq Cr^n\varepsilon^{\Theta a}\left(r^{-\alpha}(\tau+1)+\frac{1}{\lambda}\right)^{\Theta} \leq Cr^n\varepsilon^{\Theta a}\left((\tau+2)r^{-\alpha}\right)^{\Theta}\\ &\leq C(\tau+2)^{\Theta}r^{n-\Theta\alpha}\varepsilon^{\Theta a} \leq C\left(\tau^{\Theta}+1\right)r^{n-\Theta\alpha}\varepsilon^{\Theta a},\end{aligned}$$

which will be substituted into (3.26). We need λ satisfying condition

$$\frac{1}{\lambda} \leq r^{-\alpha} \Leftrightarrow \lambda \geq r^{\alpha}. \tag{3.27}$$

Choosing $\lambda_0 = r_0^{\alpha} > 0$, then inequality (3.27) holds for all $\lambda \geq \lambda_0$ because $\lambda \geq \lambda_0 = r_0^{\alpha} \geq r^{\alpha}$.
Finally, from (3.24), (3.25), and (3.26), we have

$$\begin{aligned}\left|\mathcal{V}_{\varepsilon,\lambda} \cap B_r(x)\right| &\leq C\left[\left(\tau\varepsilon^a r^{n-\alpha}\right)^{\frac{n}{n-\alpha}} + \left(\left(\tau^{\Theta}+1\right)r^{n-\Theta\alpha}\varepsilon^{\Theta a}\right)^{\frac{n}{n-\Theta\alpha}}\right]\\ &\leq C\left(\tau^{\frac{n}{n-\alpha}}\varepsilon^{\frac{na}{n-\alpha}-1} + \left(\tau^{\Theta}+1\right)^{\frac{n}{n-\Theta\alpha}}\varepsilon^{\frac{\Theta na}{n-\Theta\alpha}-1}\right)\varepsilon\left|B_r(x)\right|.\end{aligned} \tag{3.28}$$

Choosing τ_1 and τ_2, which are satisfied

$$\tau_1=\tau_2\varepsilon^c=(\tau_1+\tau_2)^{-\kappa}\varepsilon^b \Leftrightarrow \begin{cases}\tau_1=\tau_2\varepsilon^c\\ \tau_2\varepsilon^c=(\tau_1+\tau_2)^{-\kappa}\varepsilon^b\end{cases} \Leftrightarrow \begin{cases}\tau_1=\tau_2\varepsilon^c,\\ \tau_2=(1+\varepsilon^c)^{-\frac{\kappa}{1+\kappa}}\varepsilon^{\frac{b-c}{1+\kappa}}.\end{cases}$$

We need $\varepsilon>0$ small enough to ensure that τ_1 and τ_2 in $(0,1)$, which is equivalent to

$$\tau_2=(1+\varepsilon^c)^{-\frac{\kappa}{1+\kappa}}\varepsilon^{\frac{b-c}{1+\kappa}}<\varepsilon^{\frac{b-c}{1+\kappa}}<1 \text{ and } \tau_1=\tau_2\varepsilon^c<1.$$

For $b>c$, it is possible to find $\varepsilon>0$ small enough to imply that τ_1 and τ_2 in $(0,1)$. Subsequently, we obtain that

$$\tau=\tau_1+\tau_2\varepsilon^c+(\tau_1+\tau_2)^{-\kappa}\varepsilon^b=3\tau_2\varepsilon^c<3\varepsilon^{\frac{b-c}{1+\kappa}}\varepsilon^c\le C\varepsilon^{\frac{b+c\kappa}{1+\kappa}}\le C\varepsilon^{\frac{b}{1+\kappa}}. \tag{3.29}$$

Combining (3.28) and (3.29), we can rewrite

$$\begin{aligned}\left|\mathcal{V}_{\varepsilon,\lambda}\cap B_r(x)\right| &\le C\left(\varepsilon^{\frac{nb}{(n-\alpha)(1+\kappa)}}\varepsilon^{\frac{na}{n-\alpha}-1}+\left(\underbrace{3^\Theta\varepsilon^{\Theta b}}_{<1}+1\right)^{\frac{n}{n-\Theta\alpha}}\varepsilon^{\frac{\Theta na}{n-\Theta\alpha}-1}\right)\varepsilon\left|B_r(x)\right|\\ &\le C\left(\varepsilon^{\frac{n[b+(1+\kappa)a]}{(n-\alpha)(1+\kappa)}-1}+\varepsilon^{\frac{\Theta na}{n-\Theta\alpha}-1}\right)\varepsilon\left|B_r(x)\right|.\end{aligned}$$

Since $a>\dfrac{n-\Theta\alpha}{\Theta n}$, we may choose b such that $\dfrac{n[b+(1+\kappa)a]}{(n-\alpha)(1+\kappa)}>1$. One can find $\varepsilon_3>0$ such that (3.8) holds for all $\varepsilon\in(0,\varepsilon_3)$. In the remaining case, when $B_{4r}(x)\cap\partial\Omega\neq\varnothing$, there exists $x_4\in\partial\Omega$, satisfying $|x-x_4|=d(x,\partial\Omega)<4r$. Since $B_{2r}(x)\subset B_{6r}(x_4)$, we get that

$$\left|\mathcal{V}_{\varepsilon,\lambda}\cap B_r(x)\right|\le\left|\left\{\mathbf{M}_\alpha^r\left(\chi_{\Omega_{6r}(x_4)}P\right)>\varepsilon^{-a}\lambda\right\}\right|.$$

Using Assumption 1.4, we have $\kappa>0$ satisfying for all $\tau_1,\ \tau_2\in(0,1)$, we have

$$\begin{aligned}\frac{1}{|\Omega_{12}(x_4)|}\int_{\Omega_{12}(x_4)}f(u-v)\mathrm{d}z &\le \tau_1\frac{1}{|\Omega_{12}(x_4)|}\int_{\Omega_{12}(x_4)}f(u)\mathrm{d}z\\ &+\tau_2\frac{1}{|\Omega_{12}(x_4)|}\int_{\Omega_{12}(x_4)}g(u)\mathrm{d}z+C(\tau_1+\tau_2)^{-\kappa}\frac{1}{|\Omega_{12}(x_4)|}\int_{\Omega_{12}(x_4)}F\mathrm{d}z.\end{aligned}$$

By Assumption 1.3, we have $\Theta>1$ such that

$$\left(\frac{1}{|\Omega_6(x_4)|}\int_{\Omega_{6r}(x_4)}[f(v)]^\Theta\mathrm{d}z\right)^{\frac{1}{\Theta}}\le C\frac{1}{|\Omega_{12}(x_4)|}\int_{\Omega_{12r}(x_4)}[1+f(v)]\mathrm{d}z.$$

Finally, the proof of inequality (3.8) can be performed similarly to the first case. Hence, two conditions of the covering lemma are valid for all $\lambda > \lambda_0 = r_0^\alpha$ and $\varepsilon \in (0, \varepsilon_0)$ with $\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$. The proof is completed. ■

3.2. Distributional inequality and Lorentz estimates

Theorem 3.3. *For every* $\alpha \in \left[0, \frac{n}{\Theta}\right)$ *and* $a > \frac{n - \Theta\alpha}{n\Theta}$*, there exists positive constants* λ_0, ε_0, b, c *and* $C > 0$ *such that the following distribution inequality*

$$d_P^\alpha\left(\varepsilon^{-a}\lambda\right) \le C\varepsilon d_P^\alpha(\lambda) + d_F^\alpha\left(\varepsilon^b\lambda\right) + d_Q^\alpha\left(\varepsilon^c\lambda\right), \tag{3.30}$$

holds for all $\lambda > \lambda_0$ *and* $\varepsilon \in (0, \varepsilon_0)$.

Proof. Applying Theorem 3.2, for each $a > \frac{n - \Theta\alpha}{n\Theta}$, there exists positive constants λ_0, ε_0, b, c and $C > 0$ such that $|\mathcal{V}_{\varepsilon,\lambda}| \le C\varepsilon|\mathcal{W}_\lambda|$ holds for all $\lambda > \lambda_0$ and $\varepsilon \in (0, \varepsilon_0)$. We have

$$\mathcal{V}_{\varepsilon,\lambda} = \mathcal{V}_1 \cap \mathcal{V}_2 \cap \mathcal{V}_3, \text{ with } \mathcal{V}_1 = \left\{\mathbf{M}_\alpha P > \varepsilon^{-a}\lambda\right\}, \mathcal{V}_2 = \left\{\mathbf{M}_\alpha F \le \varepsilon^b\lambda\right\}, \mathcal{V}_3 = \left\{\mathbf{M}_\alpha Q \le \varepsilon^c\lambda\right\}.$$

For every subset $\mathcal{V} \subset \Omega$, we denote $\mathcal{V}^c$ by the complement of $\mathcal{V}$, i.e. $\mathcal{V}^c = \Omega \setminus \mathcal{V}$. Using several simple decompositions, we obtain that

$$|\mathcal{V}_1| = |\mathcal{V}_1 \cap \Omega| \le \left|\mathcal{V}_1 \cap \left[(\mathcal{V}_2 \cap \mathcal{V}_3) \cup (\mathcal{V}_2 \cap \mathcal{V}_3)^c\right]\right| \le |\mathcal{V}_{\varepsilon,\lambda}| + \left|(\mathcal{V}_2 \cap \mathcal{V}_3)^c\right| \le C\varepsilon|\mathcal{W}_\lambda| + |\mathcal{V}_2^c| + |\mathcal{V}_3^c|.$$

Therefore, we can rewrite the above inequality as follows

$$\left|\left\{\mathbf{M}_\alpha P > \varepsilon^{-a}\lambda\right\}\right| \le C\varepsilon\left|\left\{\mathbf{M}_\alpha P > \lambda\right\}\right| + \left|\left\{\mathbf{M}_\alpha F > \varepsilon^b\lambda\right\}\right| + \left|\left\{\mathbf{M}_\alpha Q > \varepsilon^c\lambda\right\}\right|,$$

which is the same as distributional inequality (3.30) by presenting the notion in (2.2).

Theorem 3.4. *For every* $\alpha \in \left[0, \frac{n}{\Theta}\right), 0 < q < \frac{n\Theta}{n - \Theta\alpha}$ *and* $0 < s \le \infty$*, there holds*

$$\|\mathbf{M}_\alpha P\|_{L^{q,s}(\Omega)} \le C\left(1 + \|\mathbf{M}_\alpha F\|_{L^{q,s}(\Omega)} + \|\mathbf{M}_\alpha Q\|_{L^{q,s}(\Omega)}\right). \tag{3.31}$$

Proof of Theorem 3.4. First of all, using the definition of quasi-norm in Lorentz space and changing variables from λ to $\varepsilon^{-a}\lambda$, we get that

$$\|\mathbf{M}_\alpha P\|_{L^{q,s}(\Omega)}^s = q\int_0^\infty \lambda^s \left|\left\{\mathbf{M}_\alpha P > \lambda\right\}\right|^{\frac{s}{q}} \frac{\mathrm{d}\lambda}{\lambda} = q\varepsilon^{-as}\int_0^\infty \lambda^s \left[d_P^\alpha\left(\varepsilon^{-a}\lambda\right)\right]^{\frac{s}{q}} \frac{\mathrm{d}\lambda}{\lambda}.$$

Thanks to (3.30), it implies that

$$\begin{aligned}
\|\mathbf{M}_\alpha P\|_{L^{q,s}(\Omega)}^s &= q\varepsilon^{-as}\int_0^{\lambda_0} \lambda^s \left[d_P^\alpha\left(\varepsilon^{-a}\lambda\right)\right]^{\frac{s}{q}} \frac{\mathrm{d}\lambda}{\lambda} + q\varepsilon^{-as}\int_{\lambda_0}^\infty \lambda^s \left[d_P^\alpha\left(\varepsilon^{-a}\lambda\right)\right]^{\frac{s}{q}} \frac{\mathrm{d}\lambda}{\lambda} \\
&\le q\varepsilon^{-as}\int_0^{\lambda_0} \lambda^s \left[d_P^\alpha\left(\varepsilon^{-a}\lambda\right)\right]^{\frac{s}{q}} \frac{\mathrm{d}\lambda}{\lambda} \\
&\quad + q\varepsilon^{-as}\int_{\lambda_0}^\infty \lambda^s \left(C\varepsilon d_P^\alpha(\lambda) + d_F^\alpha\left(\varepsilon^b\lambda\right) + d_Q^\alpha\left(\varepsilon^c\lambda\right)\right)^{\frac{s}{q}} \frac{\mathrm{d}\lambda}{\lambda}.
\end{aligned}$$

It follows that

$$\|\mathbf{M}_\alpha P\|^s_{L^{q,s}(\Omega)} \le q\varepsilon^{-as}\int_0^{\lambda_0}\lambda^s\left[d_P^\alpha\left(\varepsilon^{-a}\lambda\right)\right]^{\frac{s}{q}}\frac{\mathrm{d}\lambda}{\lambda}$$
$$+Cq\varepsilon^{-as}\left(\int_{\lambda_0}^\infty \lambda^s\varepsilon^{\frac{s}{q}}\left[d_P^\alpha(\lambda)\right]^{\frac{s}{q}}\frac{\mathrm{d}\lambda}{\lambda}+\int_{\lambda_0}^\infty \lambda^s\left[d_F^\alpha\left(\varepsilon^b\lambda\right)\right]^{\frac{s}{q}}\frac{\mathrm{d}\lambda}{\lambda}+\int_{\lambda_0}^\infty \lambda^s\left[d_Q^\alpha\left(\varepsilon^c\lambda\right)\right]^{\frac{s}{q}}\frac{\mathrm{d}\lambda}{\lambda}\right).$$

By performing some variable changes, it follows that

$$\|\mathbf{M}_\alpha P\|^s_{L^{q,s}(\Omega)} \le C|\Omega|^{\frac{s}{q}}\lambda_0^s + C\left(\varepsilon^{\left(\frac{1}{q}-a\right)s}\|\mathbf{M}_\alpha P\|^s_{L^{q,s}(\Omega)}+\varepsilon^{-(a+b)s}\|\mathbf{M}_\alpha F\|^s_{L^{q,s}(\Omega)}+\varepsilon^{-(a+c)s}\|\mathbf{M}_\alpha Q\|^s_{L^{q,s}(\Omega)}\right)$$
$$\le C\left(1+\varepsilon^{\frac{1}{q}-a}\|\mathbf{M}_\alpha P\|_{L^{q,s}(\Omega)}+\varepsilon^{-a-b}\|\mathbf{M}_\alpha F\|_{L^{q,s}(\Omega)}+\varepsilon^{-a-c}\|\mathbf{M}_\alpha Q\|_{L^{q,s}(\Omega)}\right)^s.$$

It is noted that it is possible to choose $\lambda_0 = \left(\mathrm{diam}(\Omega)\right)^\alpha$. Hence, we can conclude that

$$\|\mathbf{M}_\alpha P\|_{L^{q,s}(\Omega)} \le C\left(1+\varepsilon^{\frac{1}{q}-a}\|\mathbf{M}_\alpha P\|_{L^{q,s}(\Omega)}+\varepsilon^{-a-b}\|\mathbf{M}_\alpha F\|_{L^{q,s}(\Omega)}+\varepsilon^{-a-c}\|\mathbf{M}_\alpha Q\|_{L^{q,s}(\Omega)}\right). \quad (3.32)$$

For every $0<q<\dfrac{n\Theta}{n-\Theta\alpha}$, one can choose a such that $\dfrac{1}{q}-a>0$. It allows us to choose $\varepsilon>0$ small enough in (3.32) to obtain (3.31). The proof is now completed. ∎

4. Applications

In this section, we discuss how to apply the previous results to regularity theory for quasi-linear elliptic problems with variable exponents. For simplicity, we will consider a typical version of these classes of problems. Specifically, we study following *p(x)*-Laplace equation

$$-\Delta_{p(x)}u = -\mathrm{div}(|\mathbf{g}|^{p(x)-2}\,\mathbf{g}) \text{ in } \Omega,\ u=0 \text{ on } \Omega,$$

where Ω is an open bounded domain in $\mathbb{R}^n$ for $n\ge 2$ and $\mathbf{g}$ is a data function. Here, we denote by $\Delta_{p(x)}u = \mathrm{div}(|\nabla u|^{p(x)-2}\nabla u)$ the *p(x)*-Laplace operator of u. Moreover, the variable exponent $p(\cdot)$ is assumed to be continuous and satisfies the following condition

$$1<p_1\le p(x)\le p_2<\infty, \quad \text{for all} \quad x\in\Omega.$$

Tran et al. (2023) proved that for every $x_0\in\overline{\Omega}$ and $r>0$, one can find a function v and a constant $\Theta>1$ such that

$$\frac{1}{|\Omega_r(x_0)|}\int_{\Omega_r(x_0)}|\nabla v|^{\Theta p(x)}\,\mathrm{d}x \le C\left[1+\left(\frac{1}{|\Omega_{2r}(x_0)|}\int_{\Omega_{2r}(y)}|\nabla v|^{p(x)}\,\mathrm{d}x\right)^\Theta\right],$$

and $\int_{\Omega_{2r}(x_0)}|\nabla u-\nabla v|^{p(x)}\,\mathrm{d}x \le \tau\int_{\Omega_{2r}(x_0)}|\nabla u|^{p(x)}\,\mathrm{d}x + C_\tau\int_{\Omega_{2r}(x_0)}|\mathbf{g}|^{p(x)}\,\mathrm{d}x,$

for every $\tau \in (0,1)$ and for a suitable assumption on the boundary of Ω. In particular, we obtain the global estimate $\int_\Omega |\nabla u|^{p(x)} \mathrm{d}x \le C \int_\Omega |\mathbf{g}|^{p(x)} \mathrm{d}x$. Therefore, it is possible to check that Assumptions 1.1–1.4 are valid for the functions $f(u) := |\nabla u|^{p(x)}$, $g(u) := 0$ and $F(x) := |\mathbf{g}|^{p(x)}$. From Theorem 3.4, it can be concluded that $\|\mathbf{M}_\alpha f(u)\|_{L^{q,s}(\Omega)} \le C\left(1 + \|\mathbf{M}_\alpha F\|_{L^{q,s}(\Omega)}\right)$ for $\alpha \in \left[0, \frac{n}{\Theta}\right)$, $0 < q < \frac{n\Theta}{n - \Theta\alpha}$ and $0 < s \le \infty$. Using a similar technique, our distribution inequality can also be applied for non-uniformly elliptic problems with variable exponents (Tran et al., 2022, 2024).

- ***Conflict of Interest:*** *Authors have no conflict of interest to declare.*
- ***Acknowledgement:*** *This research is funded by Ho Chi Minh City University of Education Foundation for Science and Technology under grant number CS.2022.19.19TĐ.*

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DẠNG TỔNG QUÁT CỦA BẤT ĐẲNG THỨC HÀM PHÂN PHỐI VÀ ỨNG DỤNG

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TÓM TẮT

Bất đẳng thức hàm phân phối gần đây được đề xuất bởi các tác giả Trần & Nguyễn có thể sử dụng để khảo sát các đánh giá gradient cho nghiệm của phương trình đạo hàm riêng. Đặc biệt hơn, các tác giả đã đề xuất một số điều kiện đủ cho hai hàm đo được nhằm thu lại đánh giá so sánh giữa hai chuẩn của hai hàm trên không gian Lebesgue tổng quát. Các kết quả tiếp tục được ứng dụng trong một số lớp bài toán dạng p-Laplace. Trong bài báo này, chúng tôi mở rộng bất đẳng thức này để ứng dụng được trong nhiều lớp phương trình khác. Một cách chính xác hơn, bất đẳng thức hàm phân phối chúng tôi đề xuất có thể áp dụng được cho phương trình dạng p(x)-Laplace, được biết đến như là dạng phương trình tựa tuyến tính với số mũ biến.

Từ khoá: bất đẳng thức hàm phân phối; bài toán elliptic tựa tuyến tính; không gian Lorentz; lí thuyết chính quy; phương trình *p(x)*-Laplace; số mũ biến