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Research Article

CHARACTERIZING THE COMPACTNESS OF CALDERÓN-ZYGMUND COMMUTATORS OF TYPE THETA ON GENERALIZED MORREY-LORENTZ SPACES

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ABSTRACT

In this paper, we characterize the compactness of the Calderón-Zygmund commutator $[b,T]$ *of type* θ *on generalized Morrey-Lorentz spaces* $\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$. *More precisely, we prove that if* $b \in CMO(\mathbb{R}^n)$, *which is the* $BMO(\mathbb{R}^n)$ *closure of* $C_c^{\infty}(\mathbb{R}^n)$, *then* $[b,T]$ *is a compact operator on* $\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$ *for all* $1 < p < \infty$ *and* $1 \le r < \infty$.

Keywords: Calderón-Zygmund commutator of type θ; generalized Morrey-Lorentz space; compactness

1. Introduction

It is known that the theory of commutators has many important applications to some nonlinear partial differential equations. When T is a Calderón-Zygmund operator and $b \in BMO(\mathbb{R}^n)$, the L^p - compactness of [*b*, *T*] was first obtained by Uchiyama (1978). Since then, the compactness of commutators of classical operators on various function spaces has been considered. Beatrous and Li (1993) proved the boundedness and compactness characterization of $[b,T]$ on $L^p(X)$, where X is a space of homogeneous type. After that, Chen (2011) obtained the compactness of commutators for singular integrals on Morrey spaces. Later, the Lorentz boundedness and compactness of integral commutators on spaces of homogeneous type were proved by Dao and Krantz (2021). More recently, Tran et al. (2024) proved a compactness characterization of commutators of Calderón-Zygmund type in generalized Morrey-Lorentz spaces.

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On the other hand, Yabuta (1985) first introduced Calderón-Zygmund operators of type theta to facilitate his study of certain classes of pseudodifferential operators. Since then, many researchers have further studied the properties of these operators and their commutators. Liu et al. (2002) showed that if $b \in BMO\left(\mathbb{R}^n\right)$ and T is a Calderón-Zygmund operator of type theta, then $[b,T]$ is bounded from $H^1\left(\mathbb{R}^n\right)$ to weak $L^1\left(\mathbb{R}^n\right)$. Later, Thai et al. (2022) obtained the boundedness of Calderón-Zygmund operators and commutators of type theta on generalized weighted Lorentz spaces $\Lambda_u^p(w)$. Recently, Le et al. (2024) proved that the commutators $[b,T]$ of type theta are also bounded on generalized Morrey-Lorentz spaces.

For the reader's convenience, we recall below the definition of a generalized Morrey-Lorentz space and Calderón-Zygmund operator T of type theta.

Definition 1.1. *Let* $0 < p < \infty$, $0 < r \le \infty$ *and* φ *be functions satisfying the following conditions:*

$$\begin{cases} i)\ \varphi : (0;\infty) \to (0;\infty) \text{ is nonincreasing}, \\ ii)\ |B_t|\varphi^p(t) \text{ is nondecreasing, for any ball } B_t \subset X, \\ iii)\ \varphi(2t) \le D\varphi(t),\ \forall t > 0, \text{ for some constant } 0 < D < 1. \end{cases} \tag{1.1}$$

Then the generalized Morrey-Lorentz space $\mathbf{M}_{\varphi}^{p,r}\left(\mathbb{R}^n\right)$ *is defined as a set of all real-valued functions* f *with finite norm:*

$$\|f\|_{\mathbf{M}_{\varphi}^{p,r}} := \sup_{B(x,t)} \frac{\|f\|_{L^{p,r}(B(x,t))}}{|B(x,t)|^{1/p}\varphi(t)}, \tag{1.2}$$

where the supremum is taken over all balls $B(x,t)$ *in* $\mathbb{R}^n$, *and* $\|f\|_{L^{p,r}(B(x,t))}$ *denotes the Lorentz norm of* f *on* $B(x,t)$ *(see Grafakos, 2008, Definition 1.4.6).*

Definition 1.2. (Yabuta, 1985) *Let* θ *be a nonnegative, nondecreasing function on* $(0,\infty)$ *with*

$$\int_0^1 \theta(t)t^{-1}dt < \infty. \tag{1.3}$$

A continuous function $K(x,y)$ *on* $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) : x \in \mathbb{R}^n\}$ *is said to be a standard kernel of type* θ *if it satisfies the following conditions.*

(i) $$|K(x,y)| \le \frac{C}{|x-y|^n}. \tag{1.4}$$

(ii) $|K(x,y)-K(x_0,y)|+|K(y,x)-K(y,x_0)|\le C|x_0-y|^{-n}\,\theta\left(\frac{|x_0-x|}{|y-x_0|}\right),$ (1.5)

for every x, x_0, y *with* $2|x-x_0|<|y-x_0|$.

Definition 1.3. (Yabuta, 1985) *Let* θ *be a function as in Definition 1.3. A linear operator* T *from* $\mathcal{S}(\mathbb{R}^n)$ *to* $\mathcal{S}'(\mathbb{R}^n)$ *is said to be a Calderón-Zygmund operator of type* θ *if it satisfies the following conditions.*

(i) T *is bounded on* $L^2(\mathbb{R}^n)$, *which means*

$$\|Tf\|_{L^2}\le C\|f\|_{L^2} \text{ for every } f\in C_0^\infty(\mathbb{R}^n). \tag{1.6}$$

(ii) There exists a standard kernel K *of type* θ *such that for every function* $f\in C_0^\infty(\mathbb{R}^n)$ *and* $x\notin supp(f)$

$$Tf(x)=\int_{\mathbb{R}^n}K(x,y)f(y)dy. \tag{1.7}$$

Definition 1.4. *Let* T *be a Calderón-Zygmund operator of type* θ *with strong conditions in (1.7). Suppose that* $b\in BMO(\mathbb{R}^n)$, *then the commutator* $[b,T]$ *of type* θ *is defined by*

$$[b,T]f(x):=b(x)T(f)(x)-T(bf)(x) \tag{1.8}$$

for measurable functions f.

Definition 1.5. *(i) A function* $b\in L^1_{loc}(\mathbb{R}^n)$ *is said to belong to* $BMO(\mathbb{R}^n)$ *if*

$$\|b\|_{BMO(\mathbb{R}^n)}:=\sup_B\frac{1}{|B|}\int_B|b(x)-b_B|dx<\infty,$$

where $b_B=\frac{1}{|B|}\int_B b(x)dx$, *and the supremum is taken over all balls* $B\subset\mathbb{R}^n$

(ii) We denote by $CMO(\mathbb{R}^n)$ *, the* BMO *closure of* $C_c^\infty(\mathbb{R}^n)$, *where* $C_c^\infty(\mathbb{R}^n)$ *is the set of all functions in* $C^\infty(\mathbb{R}^n)$ *with compact support.*

Inspired by the above works, this study aims to study the compactness of Calderón–Zygmund commutators of type theta on generalized Morrey – Lorentz spaces in this paper. More specifically, in Section 2, we first recall a characterization of a precompact subset in $\mathbf{M}_\varphi^{p,r}(\mathbb{R}^n)$ for $1<p<\infty$ and $1\le r<\infty$ (see Lemma 2.1). Then we prove that if $b\in CMO(\mathbb{R}^n)$ then $[b,T]$ is a compact operator on $\mathbf{M}_\varphi^{p,r}(\mathbb{R}^n)$ (see Theorem 2.1).

As usual, for any $1\le q<\infty$, we denote by q' the conjugate exponent of q, that is, $\frac{1}{q}+\frac{1}{q'}=1$. We also denote a constant by C, which only depends on p,q,r,n,φ and may change on different lines. In addition, we write $A \lesssim B$ if there exists a constant $C > 0$ such that $A \le CB$ and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. Finally, we denote by B_t a ball in $\mathbb{R}^n$ with radius $t > 0$ and by 1_A the characteristic function of a subset $A \subset \mathbb{R}^n$.

2. Main results

The following lemmas are needed for proving this study's main result.

Lemma 2.1. *(Tran et al., 2024, Lemma 5.1) Let* $1 < p < \infty$, $1 \le r \le \infty$ *and* φ *satisfy (1.1). Assume that the set* $\mathcal{G}$ *in* $\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$ *satisfies the following conditions:*

(i) $\sup_{f \in \mathcal{G}} \| f \|_{M_{\varphi}^{p,r}} < \infty$,

(ii) $\lim_{y \to 0} \| f(\cdot + y) - f(\cdot) \|_{M_{\varphi}^{p,r}} = 0$ *uniformly in* $f \in \mathcal{G}$,

(iii) $\lim_{R \to \infty} \| f 1_{B_R^c} \|_{M_{\varphi}^{p,r}} = 0$ *uniformly in* $f \in \mathcal{G}$.

Then $\mathcal{G}$ *is precompact in* $\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$.

Lemma 2.2. *(*Le et al., 2024, *Theorem 2.2) Let* $1 < p < \infty$, $1 \le r \le \infty$ *and* φ *satisfy (1.1). Let* T *be a Calderón – Zygmund operator of type* θ *with* $\int_0^1 \theta(t) t^{-1} |\log t| dt < \infty$. *If* $b \in BMO(\mathbb{R}^n)$ *then* $[b,T]$ *maps* $\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n) \to \mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$. *Moreover, we have*

$$\| [b,T](f) \|_{\mathbf{M}_{\varphi}^{p,r}} \lesssim \|b\|_{BMO} \| f \|_{\mathbf{M}_{\varphi}^{p,r}},$$

for any $f \in \mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$.

It is now ready to prove the following main theorem.

Theorem 2.1. *Let* $p \in (1,\infty)$, $r \in [1,\infty]$ *and* φ *satisfy (1.1). If* $b \in CMO(\mathbb{R}^n)$, *and* T *is a Calderón – Zygmund operator of type* θ *with* $\int_0^1 \theta(t) t^{-1} |\log t| dt < \infty$, *then* $[b,T]$ *is a compact operator on* $\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$.

Proof. Assume that $b \in CMO(\mathbb{R}^n)$. Let $\mathcal{G}$ be a bounded set in $\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$. We need to show that $[b,T](\mathcal{G})$ is precompact in $\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$.

Indeed, since $b \in CMO(\mathbb{R}^n)$, then for every $\varepsilon > 0$, there exists a function $b_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ such that

$$\| b - b_\varepsilon \|_{BMO} < \varepsilon.$$

By the triangle inequality and Lemma 2.1, we have for every $f \in \mathcal{G}$

$$\begin{aligned}\left\|[b,T](f)\right\|_{M_{\varphi}^{p,r}} &\le \left\|[b-b_{\varepsilon},T](f)\right\|_{M_{\varphi}^{p,r}}+\left\|[b_{\varepsilon},T](f)\right\|_{M_{\varphi}^{p,r}}\\ &\lesssim \|b-b_{\varepsilon}\|_{BMO}\|f\|_{M_{\varphi}^{p,r}}+\left\|[b_{\varepsilon},T](f)\right\|_{M_{\varphi}^{p,r}}\\ &\lesssim C\varepsilon+\left\|[b_{\varepsilon},T](f)\right\|_{M_{\varphi}^{p,r}}.\end{aligned}$$

With this inequality in mind, it suffices to show that $[b_{\varepsilon},T](\mathcal{G})$ is precompact in $\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$ for a given $\varepsilon>0$ small enough.

Since $\mathcal{G}$ is a bounded set in $\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$, it follows from Lemma 2.2 that $[b_{\varepsilon},T](\mathcal{G})$ satisfies *(i)* of Lemma 2.1.

Next, we show that $[b_{\varepsilon},T](\mathcal{G})$ also satisfies *(iii)* of Lemma 2.1. Indeed, suppose that $\operatorname{supp}(b_{\varepsilon})\subset B_{R_{\varepsilon}}$, for some $R_{\varepsilon}>10$. Then, for any $f\in\mathcal{G}$ and for $x\in B_R^c$ with $R>10R_{\varepsilon}$, we observed that $|x-y|\approx|x|$ for any $y\in B_{R_{\varepsilon}}$.

Thus, for any $x\in B_R^c$ we have

$$\begin{aligned}\left|[b_{\varepsilon},T](f)(x)\right|=\left|T(b_{\varepsilon}f)(x)\right| &\le \|b_{\varepsilon}\|_{L^{\infty}}\int_{B(0,R_{\varepsilon})}|K(x,y)||f(y)|dy\\ &\le \|b_{\varepsilon}\|_{L^{\infty}}\int_{B(0,R_{\varepsilon})}\frac{C}{|x-y|^n}|f(y)|dy\\ &\le \|b_{\varepsilon}\|_{L^{\infty}}\int_{B(0,R_{\varepsilon})}\frac{C}{|R-R_{\varepsilon}|^n}|f(y)|dy\\ &\le \frac{\|b_{\varepsilon}\|_{L^{\infty}}}{\left(\frac{1}{2}R\right)^n}\int_{B(x,R_{\varepsilon})}|f(x-w)|dw.\end{aligned}\tag{2.10}$$

For every ball $B_t=B(x_0,t)$ in $\mathbb{R}^n$, by (2.10), and Minkowski's inequality, we obtain:

$$\frac{\left\|[b_{\varepsilon},T](f)1_{B_R^c}\right\|_{L^{p,r}(B_t)}}{|B_t|^{\frac{1}{p}}\varphi(t)}\lesssim\frac{\|b_{\varepsilon}\|_{L^{\infty}}}{|B_t|^{\frac{1}{p}}\varphi(t)\left(\frac{1}{2}R\right)^n}\int_{B(x,R_{\varepsilon})}\|f(\cdot-w)\|_{L^{p,r}(B_t)}dw$$

$$\lesssim\frac{\|b_{\varepsilon}\|_{L^{\infty}}}{\left(\frac{1}{2}R\right)^n}\int_{B(x,R_{\varepsilon})}\frac{\|f\|_{L^{p,r}(B(x_0-w,t))}}{|B(x_0-w,t)|^{\frac{1}{p}}\varphi(t)}dw\lesssim\frac{\|b_{\varepsilon}\|_{L^{\infty}}}{\left(\frac{1}{2}R\right)^n}\int_{B(x,R_{\varepsilon})}\|f\|_{M_{\varphi}^{p,r}}dw$$

$$\lesssim \|b_\varepsilon\|_{L^\infty} \frac{|B(x,R_\varepsilon)|}{\left(\frac{1}{2}R\right)^n} \|f\|_{M_\varphi^{p,r}} \lesssim \|b_\varepsilon\|_{L^\infty} \frac{|B(x,R_\varepsilon)|}{\left(\frac{1}{2}R\right)^n},$$

uniformly in $f \in \mathcal{G}$.

This implies that

$$\left\| [b_\varepsilon, T](f) 1_{B_R^c} \right\|_{M_\varphi^{p,r}} \lesssim \|b_\varepsilon\|_{L^\infty} \frac{|B(x,R_\varepsilon)|}{\left(\frac{1}{2}R\right)^n}, \forall f \in \mathcal{G}.$$

Thus, $\left\| [b_\varepsilon, T](f) 1_{B_R^c} \right\|_{M_\varphi^{p,r}} \to 0$ when $R \to \infty$ uniformly in $f \in \mathcal{G}$. In other words, $[b_\varepsilon, T](\mathcal{G})$ verifies *(iii)* of Lemma 2.1.

It remains to prove the equicontinuity of $[b_\varepsilon, T]$, that is, (*ii*) of Lemma 2.1 holds. In fact, we show that for every $\delta > 0$, if $|z|$ is sufficiently small (merely depending on δ), then

$$\left\| [b_\varepsilon, T](f)(\cdot + z) - [b_\varepsilon, T](f)(\cdot) \right\|_{M_\varphi^{p,r}} \le C\delta^n, \tag{2.11}$$

uniformly in $f \in \mathcal{G}$, where the constant $C > 0$ is independent of $f, \delta, |z|$.

To obtain the desired result, we recall the maximal operator of T, defined by

$$\mathcal{T}(f)(x) = \sup_{\tau > 0} \left[T_\tau(f)(x) \right], \tag{2.12}$$

where T_τ, the truncated operator of T, is given by

$$T_\tau(f)(x) = \int_{\{|x-y|>\tau\}} K(x,y) f(y) dy. \tag{2.13}$$

Now, for any $x \in \mathbb{R}^n$, we express

$$[b_\varepsilon, T](f)(x+z) - [b_\varepsilon, T](f)(x) = \int_{\mathbb{R}^n} \left[b_\varepsilon(y) - b_\varepsilon(x+z) \right] K(x+z,y) f(y) dy$$

$$- \int_{\mathbb{R}^n} \left[b_\varepsilon(y) - b_\varepsilon(x) \right] K(x,y) f(y) dy = \int_{|x-y|>\delta^{-1}|z|} \left[b_\varepsilon(x) - b_\varepsilon(x+z) \right] K(x,y) f(y) dy$$

$$+ \int_{|x-y|>\delta^{-1}|z|} \left[b_\varepsilon(y) - b_\varepsilon(x+z) \right] \left[K(x+z,y) - K(x,y) \right] f(y) dy$$

$$+ \int_{|x-y|\le\delta^{-1}z} \left[b_\varepsilon(x) - b_\varepsilon(y) \right] K(x,y) f(y) dy + \int_{|x-y|\le\delta^{-1}z} \left[b_\varepsilon(y) - b_\varepsilon(x+z) \right] K(x+z,y) f(y) dy$$

$$:= \mathbf{I}_1 + \mathbf{I}_2 + \mathbf{I}_3 + \mathbf{I}_4.$$

We first estimate $\mathbf{I}_1$.

$$\begin{aligned}|\mathbf{I_1}| &\leq |b_\varepsilon(x)-b_\varepsilon(x+z)|\left|\int_{|x-y|>\delta^{-1}|z|} K(x,y)f(y)dy\right| \\ &\leq |b_\varepsilon(x+z)-b_\varepsilon(x)|\mathcal{T}(f)(x).\end{aligned}$$

Since b_ε is uniformly continuous on $\mathbb{R}^n$, then we deduce from the last inequality that

$|\mathbf{I_1}| \leq \delta\mathcal{T}(f)(x)$, as $|z| \to 0$.

We recall here Cotlar's inequality (see Torchinsky, 1986, Lemma 6.1). That is for all $q>0$:

$$\mathcal{T}(f)(x) \leq C\left[\mathcal{M}_q(T(f))(x)+\mathcal{M}(f)(x)\right], \tag{2.14}$$

where $\mathcal{M}_q(T(f))(x)$, $x\in\mathbb{R}^n$ is given by

$$\mathcal{M}_q(T(f))(x) = \sup_{x\in B}\left\{\frac{1}{|B|^{\frac{1}{q}}}\|T(f)\|_{L^q}\right\}.$$

Applying Cotlar's inequality and Lemma 2.2 yields :

$$\|\mathbf{I_1}\|_{M_\varphi^{p,r}} \leq \delta\|\mathcal{T}(f)\|_{M_\varphi^{p,r}} \lesssim \delta\|f\|_{M_\varphi^{p,r}}. \tag{2.15}$$

For $\mathbf{I}_2$, we use the smoothness of kernel K and the doubling property of Lebesgue measure to get

$$\begin{aligned}|\mathbf{I_2}| &\lesssim \|b_\varepsilon\|_{L^\infty}\int_{|x-y|>\delta^{-1}|z|}|K(x+z,y)-K(x,y)||f(y)|dy \\ &= \|b_\varepsilon\|_{L^\infty}\sum_{k\geq 0}\frac{\delta^n}{|z|^n}\int_{D_{k+1}\setminus D_k}\theta\left(\frac{|z|}{|y-x|}\right)|f(y)|dy \\ &\leq \|b_\varepsilon\|_{L^\infty}\frac{\delta^n}{|z|^n}\sum_{k\geq 0}\int_{D_{k+1}\setminus D_k}\theta\left(\delta 2^{-k}\right)|f(y)|dy \\ &\leq \|b_\varepsilon\|_{L^\infty}\delta^n\sum_{k\geq 0}\theta\left(\delta 2^{-k}\right)\frac{1}{|z|^n}\int_{D_{k+1}}|f(y)|dy \\ &\lesssim \|b_\varepsilon\|_{L^\infty}\delta^n\sum_{k\geq 0}\theta\left(\delta 2^{-k}\right)\mathcal{M}(f)(x) \\ &\lesssim \|b_\varepsilon\|_{L^\infty}\delta^n\int_0^1\frac{\theta(\delta t)}{t}dt\,\mathcal{M}(f)(x) \\ &\lesssim \|b_\varepsilon\|_{L^\infty}\delta^n\mathcal{M}(f)(x),\end{aligned}$$

where $D_k = B\left(x, 2^k\delta^{-1}|z|\right)$, $k\geq 0$.

Then we deduce that from the last inequality that:

$$\|\mathbf{I_2}\|_{M_\varphi^{p,r}} \lesssim \delta^n\|b_\varepsilon\|_{L^\infty}\|\mathcal{M}(f)\|_{M_\varphi^{p,r}} \lesssim \delta^n\|b_\varepsilon\|_{L^\infty}\|f\|_{M_\varphi^{p,r}}. \tag{2.16}$$

Next, we estimate $\mathbf{I}_3$. For any $k \geq 0$, let us set $B_k = B\left(x, \delta^{-1}2^{-k}|z|\right)$. Then, it follows from the size condition of K that

$$\begin{aligned}
|\mathbf{I}_3| &\leq C\|\nabla b_\varepsilon\|_{L^\infty} \int_{|x-y|\leq\delta^{-1}|z|} \frac{|x-y|}{|B(x,|x-y|)|}|f(y)|dy \\
&\lesssim \|\nabla b_\varepsilon\|_{L^\infty} \sum_{k\geq 0} \frac{2^{-k}\delta^{-1}|z|}{|B_{k+1}|} \int_{B_k\setminus B_{k+1}} |f(y)|dy \\
&\leq \delta^{-1}|z|\|\nabla b_\varepsilon\|_{L^\infty} \sum_{k\geq 0} 2^{-k}\frac{|B_k|}{|B_{k+1}|}\frac{1}{|B_k|}\int_{B_k}|f(y)|dy \\
&\leq \delta^{-1}|z|\|\nabla b_\varepsilon\|_{L^\infty} \sum_{k\geq 0} C_\mu 2^{-k}\mathcal{M}(f)(x) \\
&\lesssim \delta\|\nabla b_\varepsilon\|_{L^\infty}\mathcal{M}(f)(x), \text{ provided that } |z| < \delta^2.
\end{aligned}$$

With the last inequality in mind, it follows from the $\mathbf{M}_\varphi^{p,r}$ - bound of the operator $\mathcal{M}$ that

$$\|\mathbf{I}_3\|_{M_\varphi^{p,r}} \lesssim \delta\|\nabla b_\varepsilon\|_{L^\infty}\|\mathcal{M}(f)\|_{M_\varphi^{p,r}} \lesssim \delta\|\nabla b_\varepsilon\|_{L^\infty}\|f\|_{M_\varphi^{p,r}}. \tag{2.17}$$

Finally, we treat $\mathbf{I}_4$. Since $\operatorname{supp}(b_\varepsilon) \subset B(0,R_\varepsilon)$, then it is sufficient to consider $x \in B(0,2R_\varepsilon)$ when $z \to 0$.

Thanks to the triangle inequality, we get

$$|x-y+z| \leq |x-y| + |z| \leq 2\delta,$$

when $z \to 0$, for all $|x-y| < \delta$.

Then,

$$\begin{aligned}
|\mathbf{I}_4| &\leq C\|\nabla b_\varepsilon\|_{L^\infty} \int_{|x-y|<\delta^{-1}|z|} \frac{|x-y+z|}{|B(x+z,|x-y+z|)|}|f(y)|dy \\
&\leq C\|\nabla b_\varepsilon\|_{L^\infty} \int_{|x-y+z|<2\delta} \frac{|x-y+z|}{|B(x+z,|x-y+z|)|}|f(y)|dy.
\end{aligned}$$

By arguing as in $\mathbf{I}_3$, we also obtain

$$\|\mathbf{I}_4\|_{M_\varphi^{p,r}} \lesssim \delta\|\nabla b_\varepsilon\|_{L^\infty}\|f\|_{M_\varphi^{p,r}}. \tag{2.18}$$

For any $k \geq 0$, let us set $B_k = B\left(x+z, 2^{-k+1}|z|\right)$. Combining (2.15), (2.16), (2.17), and (2.18) yields

$$\|[b_\varepsilon,T](f)(x+z) - [b_\varepsilon,T](f)(x)\|_{M_\varphi^{p,r}} \lesssim \delta\|\nabla b_\varepsilon\|_{L^\infty}\|f\|_{M_\varphi^{p,r}}$$

uniformly in $f \in \mathcal{G}$. Therefore, $[b_\varepsilon, T]$ satisfies (*ii*) of Lemma 2.1.

Finally, from Lemma 2.1, we conclude that $[b,T]$ is a compact operator on $\mathbf{M}_\varphi^{p,r}(\mathbb{R}^n)$.

3. Conclusion

Main Result. *Let* $1 < p < \infty$, $1 \le r \le \infty$, *and* φ *satisfy (1.1). If* $b \in CMO\left(\mathbb{R}^n\right)$, *and* T *is a Calderón – Zygmund operator of type* θ *with* $\int_0^1 \theta(t) t^{-1} \left|\log t\right| dt < \infty$, *then* $\left[b, T\right]$ *is a compact operator on* $\mathbf{M}_{\varphi}^{p,r}\left(\mathbb{R}^n\right)$.

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TÍNH COMPACT CỦA HOÁN TỬ CALDERÓN-ZYGMUND LOẠI THETA TRÊN KHÔNG GIAN MORREY-LORENTZ TỔNG QUÁT

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TÓM TẮT

Trong bài báo này, chúng tôi xét tính compact của hoán tử Calderón-Zygmund $[b,T]$ *loại* θ *trong không gian Morrey – Lorentz tổng quát* $\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$. *Cụ thể hơn, chúng tôi chứng minh rằng nếu* $b \in CMO(\mathbb{R}^n)$*-bao đóng của* $BMO(\mathbb{R}^n)$ *trong* $C_c^{\infty}(\mathbb{R}^n)$ *thì* $[b,T]$ *là toán tử compact trong* $\mathbf{M}_{\varphi}^{p,r}(\mathbb{R}^n)$ *với mọi* $1 < p < \infty$ *và* $1 \le r < \infty$.

Từ khóa: hoán tử Calderón-Zygmund loại θ; không gian Morrey – Lorentz tổng quát; tính compact