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Research Article

DISTRIBUTION INEQUALITY FOR A CLASS OF QUASI-LINEAR ELLIPTIC EQUATIONS WITH MIXED DATA

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ABSTRACT

The problem of regularity for partial differential equations has been studied by many mathematicians in recent years using many different methods. With the development of harmonic analysis, Calderón-Zygmund theory plays an important role in investigating the regularity problem. In this paper, we establish Calderón-Zygmund type estimates for weak solutions to a class of quasi-linear elliptic equations with mixed data in the generalized Lorentz space. Our study is an extension of the function space to some gradient estimates in several previous papers. This result once again confirms the effectiveness of the method of using the distribution inequality on the level sets to the regularity problem for partial differential equations.

Keywords: Calderón-Zygmund estimates; distribution inequality; fractional maximal operators; Generalized Lorentz spaces; quasi-linear elliptic equations

1. Introduction

Let $p \in (1, n)$ and Ω be an open bounded domain in $\mathbb{R}^n$ with $n \geq 2$. We assume that

$$\mathbf{F} \in \left(L^p(\Omega)\right)^n, \ g \in W^{1,p}(\Omega) \text{ and } f \in L^{p'}(\Omega) \ \text{ with } p' := \frac{p}{p-1}. \tag{1.1}$$

Moreover, we consider two Carathéodory operators $\mathbb{A}, \mathbb{B} : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ that satisfy the following conditions:

$$\begin{cases} |\mathbb{A}(x,z)| + |\mathbb{B}(x,z)| \leq \kappa_1 |z|^{p-1} \\ \left(\mathbb{A}(x,z_1) - \mathbb{A}(x,z_2)\right) \cdot (z_1 - z_2) \geq \kappa_2 \left(|z_1|^2 + |z_2|^2\right)^{\frac{p-2}{2}} |z_1 - z_2|^2, \end{cases} \tag{1.2}$$

for $x \in \Omega$ a.e. and $z_1, z_2 \in \mathbb{R}^n \setminus \{0\}$, where κ_1, κ_2 are constants. The main goal of the article is to study the regularity of the following quasi-linear problem

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$$\begin{cases} -\mathrm{div}(\mathbb{A}(x,\nabla u)) = f - \mathrm{div}(\mathbb{B}(x,\mathbf{F})) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

A function $u \in g + W_0^{1,p}(\Omega)$ is called the weak solution to (1.3) if the following variational formula

$$\int_\Omega \mathbb{A}(x,\nabla u)\cdot\nabla\psi dx = -\int_\Omega f\psi dx + \int_\Omega \mathbb{B}(x,\mathbf{F})\cdot\nabla\psi dx,$$

holds for all $\psi \in W_0^{1,p}(\Omega)$. A typical form of (1.3) is the *p*-Laplace equation, and the existence result of weak solutions to such an equation can be found in Lieberman (1984) or Tolksdorff (1984).

The quasi-linear elliptic equation (1.3) has been extensively studied in the literature, including by Caffarelli and Peral (1998), DiBenedetto and Manfredi (1993), Evans (1982), Iwaniec (1983), and Uhlenbeck (1977). Moreover, the regularity for (1.3) is investigated in several cases, which depend on the form of the right-hand side. For example, when the right-hand side has divergence form, relevant results have been established by Breit et al. (2017), Byun and Wang (2004, 2008), Milakis and Toro (2010), Tran and Nguyen (2020a, 2023), and Nguyen et al. (2021). For non-divergence and measure data problems, numerous results have been established in the literature (Duzaar & Mingione, 2010, 2011; Mingione, 2010; Tran, 2019; Tran & Nguyen, 2020b, 2022). Motivated by these works, we continue to investigate the regularity of quasilinear equations with mixed data, which has been studied by Lee and Ok (2019) in Lebesgue spaces, Nguyen and Tran (2020) in Lorentz spaces and Tran et al. (2024c) in Lorentz-Morrey spaces. More precisely, we extend these results to the generalized Lorentz space. In particular, we prove the following gradient estimate

$$\|M_\alpha(|\nabla u|^p)\|_{\mathcal{L}^{s,t}_{\varphi,\mu}(\Omega)} \le C\ \|M_\alpha(|\mathbf{F}|^p + |f|^{p'} + |\nabla g|^p)\|_{\mathcal{L}^{s,t}_{\varphi,\mu}(\Omega)},$$

where M_α denotes the fractional maximal operator and $\mathcal{L}^{s,t}_{\varphi,\mu}(\Omega)$ is the generalized Lorentz space with weights. Our method is to use the distribution inequality on the level sets via fractional maximal operators. This method was proposed by Tran and Nguyen (2019, 2021).

The rest of our paper will be organized as follows. In the next section, we recall some preliminaries about Reifenberg domains, BMO quasi-norms, function spaces, Muckenhoupt weights, maximal operators, and the covering lemma. These are definitions and well-known results that can be found in many references. In the last section, we present two main results of the paper. The first one in Theorem 3.2 is the distribution function inequality on the level sets. The remaining result in Theorem 3.3 is to evaluate the global gradient estimate on the generalized Lorentz space.

2. Preliminaries

In this paper, the open ball in $\mathbb{R}^n$ of radius $r > 0$ and center ξ will be denoted by

$$B(\xi,r) = \left\{z \in \mathbb{R}^n : |z-\xi| < r\right\}.$$

We denote by $|E|$ the Lebesgue measure of E in $\mathbb{R}^n$. The diameter of Ω is defined by

$$\mathrm{diam}(\Omega) = \sup\left\{\left|\xi_1 - \xi_2\right| : \ \xi_1, \xi_2 \in \Omega\right\}.$$

The set $\{x \in \Omega : f(x) > \lambda\}$ will be simplify denoted by $\{f > \lambda\}$. Besides, the symbol C is often used to refer to constants that depend only on parameters in the data, and it is usually different after each evaluation. In some special cases, the constant C depends on certain parameters enclosed in parentheses.

Definition 2.1. (Reifenberg domain) *Let $\delta \in (0,1/8)$ and $r_0 > 0$. We say that Ω is (δ, r_0)-Reifenberg if for every $\xi_0 \in \partial\Omega$ and $r \in (0, r_0]$, there exists a new coordinate system $\{y_1, y_2, \ldots, y_n\}$ with origin at ξ_0 such that*

$$B(\xi_0, r) \cap \{y_n > \delta r\} \subset B(\xi_0, r) \cap \Omega \subset B(\xi_0, r) \cap \{y_n > -\delta r\},$$

where the set $\{(y_1, y_2, \ldots, y_n) : y_n > c\}$ is denoted by $\{y_n > c\}$.

Definition 2.2. (BMO quasi-norm) *Let $r_0 > 0$, we define*

$$[\mathbb{A}]_{r_0} = \sup_{y \in \mathbb{R}^n, 0 < r \le r_0} \frac{1}{|B(y,r)|} \int_{B(y,r)} \sup_{z \in \mathbb{R}^n \setminus \{0\}} \frac{\left|\mathbb{A}(x,z) - \overline{\mathbb{A}}_{B(y,r)}(z)\right|}{|z|^{p-1}} dx,$$

where $\overline{\mathbb{A}}_{B(y,r)}(z) = \dfrac{1}{|B(y,r)|} \displaystyle\int_{B(y,r)} \mathbb{A}(x,z) dx$. We will denote by $(\Omega, \mathbb{A}) \in (\mathcal{H})_{\delta, r_0}$ if Ω is (δ, r_0)-Reifenberg and $[\mathbb{A}]_{r_0} \le \delta$.

Definition 2.3. (Muckenhoupt weights) *Given $1 \le q < \infty$ and a non-negative weight $\mu \in L^1_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^+)$. We write $\mu \in A_q$ if $[\mu]_{A_q} < \infty$, where*

$$[\mu]_{A_q} = \begin{cases} \displaystyle\sup_{B(x,r) \subset \mathbb{R}^n} \frac{1}{|B(x,r)|} \left(\sup_{y \in B(x,r)} \frac{1}{\mu(y)} \right) \left(\int_{B(x,r)} \mu(y) dy \right), & \text{if } q = 1, \\ \displaystyle\sup_{B(x,r) \subset \mathbb{R}^n} \frac{1}{|B(x,r)|^q} \left(\int_{B(x,r)} \mu(y)^{-\frac{1}{q-1}} dy \right)^{q-1} \left(\int_{B(x,r)} \mu(y) dy \right), & \text{if } q > 1. \end{cases}$$

The Muckenhoupt weights A_∞ is defined by $A_\infty := \bigcup_{q \ge 1} A_q$.

For each $\mu \in A_\infty$ and a measurable set E in $\mathbb{R}^n$, we will denote $\mu(E) = \int_E \mu(x) dx$. Moreover, it is well-known that (see Tran et al. 2024a, 2024b), there exist some constants $\sigma_1, \sigma_2, \beta_1, \beta_2 > 0$ such that

$$\sigma_1 \left(\frac{|E|}{|B|} \right)^{\beta_1} \mu(B) \le \mu(E) \le \sigma_2 \left(\frac{|E|}{|B|} \right)^{\beta_2} \mu(B), \qquad (2.1)$$

for all balls B in $\mathbb{R}^n$ and $E \subset B$. For this reason, we denote $[\mu]_{A_\infty} = (\sigma_1, \sigma_2, \beta_1, \beta_2)$.

Definition 2.4. (Generalized Lorentz spaces) *Let* $\mu \in A_\infty$ *and* $\varphi \in L^1_{\text{loc}}\left(\mathbb{R}^+;\mathbb{R}^+\right)$*. We consider a function* Φ *defined by*

$$\Phi(\lambda) = \int_0^\lambda \varphi(\tau)d\tau, \quad 0 \le \lambda < \infty. \tag{2.2}$$

Given $s \in (0,\infty)$ *and* $0 < t \le \infty$ *, the generalized Lorentz spaces* $\mathcal{L}^{s,t}_{\varphi,\mu}(\Omega)$ *is the set of all functions* f *satisfying* $\|f\|_{\mathcal{L}^{s,t}_{\varphi,\mu}(\Omega)} < \infty$*, where*

$$\|f\|_{\mathcal{L}^{s,t}_{\varphi,\mu}(\Omega)} := \begin{cases} \left[s\int_0^\infty \lambda^{t-1}[\Phi(\mu(\{|f|>\lambda\}))]^{\frac{t}{s}} d\lambda \right]^{\frac{1}{t}}, & \text{if } t<\infty, \\ \sup_{\lambda>0} \lambda[\Phi(\mu(\{|f|>\lambda\}))]^{\frac{1}{s}}, & \text{if } t=\infty. \end{cases}$$

We refer the reader to Carro et al. (2007) for several non-trivial examples of the function Φ in (2.2) satisfying the doubling condition (3.16) in Theorem 3.3.

Definition 2.5. (Maximal operators) *For* $\alpha \in [0,n]$*, the fractional maximal operator* M_α *of a function* $h \in L^1_{\text{loc}}\left(\mathbb{R}^n\right)$ *is given by:*

$$M_\alpha f(x) = \sup_{r>0} r^{\alpha-n}\int_{B(x,r)} |f(y)|\,dy, \quad x \in \mathbb{R}^n.$$

We remark that if $\alpha = 0$*,* M_0 *is exactly the Hardy-Littlewood operator* M*. Moreover, if* $M_\alpha f(x) \le \lambda$ *for some* $x \in \Omega$ *and* $\lambda > 0$*, then*

$$\int_{B(x,r)} |f(y)|\,dy \le r^{n-\alpha}\lambda, \quad \text{for all } r>0. \tag{2.3}$$

Lemma 2.6. (Boundedness of M_α, see Tran and Nguyen (2020a)) *Let* $0 \le \alpha < n$*, one has*

$$\left|\left\{x \in \mathbb{R}^n : M_\alpha f(x) > \lambda\right\}\right| \le C(n,\alpha)\left(\frac{1}{\lambda}\int_{\mathbb{R}^n} |f(x)|\,dx\right)^{\frac{n}{n-\alpha}},$$

for all $f \in L^1\left(\mathbb{R}^n\right)$ *and* $\lambda > 0$.

Lemma 2.7. (Covering Lemma, see Caffarelli and Peral (1998)) *Let* $\varepsilon \in (0,1)$*,* $r_0 > 0$ *and* $\mu \in A_\infty$*. Assume that two measurable sets* $\mathcal{V} \subset \mathcal{W} \subset \Omega$ *satisfy* $\mu(\mathcal{V}) \le \varepsilon\mu\left(B\left(0,r_0\right)\right)$*. Suppose further that the following statement*

$$\mu(\mathcal{V} \cap B(\xi,\varrho)) > \varepsilon\mu(B(\xi,\varrho)) \Rightarrow \Omega \cap B(\xi,\varrho) \subset \mathcal{W}.$$

holds for all $\xi \in \Omega$ *and* $\varrho \in \left(0,r_0\right]$*. Then, there exists a constant* $C > 0$ *such that*

$$\mu(\mathcal{V}) \le C\varepsilon\mu(\mathcal{W}).$$

3. Main results

From now on, we will always consider u as a weak solution to (1.3) with the data in (1.1) under conditions (1.2). For simplicity of notation, let us denote by

$$|\mathcal{G}|^p = |\mathbf{F}|^p + |f|^{p'} + |\nabla g|^p .$$

Moreover, we consider $\alpha \in [0,n)$ and a Muckenhoupt weight $\mu \in A_\infty$ with

$$[\mu]_{A_\infty} = (\sigma_1, \sigma_2, \beta_1, \beta_2).$$

We will denote data $= (n, p, \kappa_1, \kappa_2, \sigma_1, \sigma_2, \beta_1, \beta_2, \alpha, \mathrm{diam}(\Omega)/r_0)$.

The proof of the following lemma can be found in Nguyen and Tran (2020).

Lemma 3.1. *There exists a positive constant $C = C(n, p, \kappa_1, \kappa_2)$ such that*

$$\int_\Omega |\nabla u|^p \, dx \le C \int_\Omega |\mathcal{G}|^p \, dx. \tag{3.1}$$

For every $\xi \in \overline{\Omega}$ and $\varrho > 0$, there exist

$$v \in W^{1,p}(\Omega \cap B(\xi,\varrho)) \cap W^{1,\infty}(\Omega \cap B(\xi, \varrho/2))$$

and a constant $\tilde{p} \in (0,p)$ such that

$$\|\nabla v\|^p_{L^\infty(\Omega \cap B(\xi,\varrho/2))} \le C\varrho^{-n} \left(\int_{\Omega \cap B(\xi,2\varrho)} |\nabla u|^p \, dx + \int_{\Omega \cap B(\xi,2\varrho)} |\mathcal{G}|^p \, dx \right),$$

and

$$\int_{\Omega \cap B(\xi,\varrho)} |\nabla u - \nabla v|^p \, dx \le C \left(\delta^p \int_{\Omega \cap B(\xi,2\varrho)} |\nabla u|^p \, dx + \delta^{\tilde{p}} \int_{\Omega \cap B(\xi,2\varrho)} |\mathcal{G}|^p \, dx \right), \text{ provided}$$

$(\Omega, \mathbb{A}) \in (\mathcal{H})_{\delta, r_0}$ for some $\delta \in (0,1/8)$ and $r_0 > 0$.

Let us introduce two distribution functions (see Grafakos (2004) for more information about the distribution function) as follows

$$d_u^\mu(\lambda) = \mu(\{M_\alpha(|\nabla u|^p) > \lambda\}), \quad d_{\mathcal{G}}^\mu(\lambda) = \mu(\{M_\alpha(|\mathcal{G}|^p) > \lambda\}) \tag{3.2}$$

When $\mu \equiv 1$, we simply denote d_u^μ by d_u.

Theorem 3.2. *Let $r_0 > 0$ and $\varepsilon \in (0,1)$. Then, one can find some constants $a = a(data) > 1$, $\delta = \delta(data, \varepsilon) \in (0,1/8)$, and $b_\varepsilon = b_\varepsilon(data, \varepsilon) \in (0,1)$ such that if $(\Omega, \mathbb{A}) \in (\mathcal{H})_{\delta, r_0}$ then the following distribution inequality*

$$d_u^\mu(a\lambda) \le C\varepsilon d_u^\mu(\lambda) + d_{\mathcal{G}}^\mu(b_\varepsilon \lambda) \tag{3.3}$$

holds for all $\lambda \in \mathbb{R}^+$. Here, the constant C depends on the data.

Proof. Let us consider the following two subsets of Ω:

$$\mathcal{V} := \left\{ M_\alpha(|\nabla u|^p) > a\lambda, M_\alpha(|\mathcal{G}|^p) \le b_\varepsilon \lambda \right\}, \text{ and } \mathcal{W} := \left\{ M_\alpha(|\nabla u|^p) > \lambda \right\},$$

where a and b_ε will be determined later. By using Lemma 2.7 in the previous section, we will show that

$$\mu(\mathcal{V}) \le C\varepsilon \mu(\mathcal{W}). \tag{3.4}$$

It is obvious that (3.4) holds if $\mathcal{V} = \varnothing$. So, we may assume that $\mathcal{V}$ is not empty, which guarantees the existence of $\xi_1 \in \Omega$ satisfying $M_\alpha(|\mathcal{G}|^p)(\xi_1) \le b_\varepsilon \lambda$. Thanks to (2.3), it implies to

$$\int_{\Omega} |\mathcal{G}|^p \, dx \le \int_{B(\xi_1, 2\mathrm{diam}(\Omega))} |\mathcal{G}|^p \, dx \le \left(2\mathrm{diam}(\Omega)\right)^{n-\alpha} b_\varepsilon \lambda. \tag{3.5}$$

Thanks to Lemma 2.6 and (3.1) in Lemma 3.1, one gets that

$$d_u(a\lambda) \le C\left(\frac{1}{a\lambda}\int_{\Omega} |\nabla u|^p \, dx\right)^{\frac{n}{n-\alpha}} \le C\left(\frac{1}{a\lambda}\int_{\Omega} |\mathcal{G}|^p \, dx\right)^{\frac{n}{n-\alpha}},$$

which from (3.5) leads to

$$d_u(a\lambda) \le C\left(\frac{1}{a\lambda}\left(2\mathrm{diam}(\Omega)\right)^{n-\alpha} b_\varepsilon \lambda\right)^{\frac{n}{n-\alpha}} \le C\left(\frac{b_\varepsilon}{a}\right)^{\frac{n}{n-\alpha}} \left(\mathrm{diam}(\Omega)\right)^n.$$

It follows that

$$d_u(a\lambda) \le C\left(\frac{\mathrm{diam}(\Omega)}{r_0}\right)^n \left(\frac{b_\varepsilon}{a}\right)^{\frac{n}{n-\alpha}} |B(0, r_0)|.$$

We can find a constant $m = m(\mathrm{diam}(\Omega)/r_0) > 1$ large enough such that

$$\left\{M_\alpha(|\nabla u|^p) > \lambda\right\} \subset \Omega \subset B(0, mr_0).$$

Since $\mu \in A_\infty$ with $[\mu]_{A_\infty} = (\sigma_1, \sigma_2, \beta_1, \beta_2)$, thanks to (2.1), there holds

$$\begin{aligned} d_u^\mu(a\lambda) &\le \sigma_2\left(\frac{d_u(a\lambda)}{|B(0, mr_0)|}\right)^{\beta_2} \mu(B(0, mr_0)) \\ &\le C\sigma_1\sigma_2\left(\frac{\mathrm{diam}(\Omega)}{r_0}\right)^{n\beta_2} \left(\frac{b_\varepsilon}{a}\right)^{\frac{n\beta_2}{n-\alpha}} \left(\frac{|B(0, mr_0)|}{|B(0, r_0)|}\right)^{-\beta_1-\beta_2} \mu(B(0, r_0)) \\ &\le C_1\left(\frac{b_\varepsilon}{a}\right)^{\frac{n\beta_2}{n-\alpha}} \mu(B(0, r_0)). \end{aligned} \tag{3.6}$$

Here, we remark that the constant C_1 still depends on the ratio $\mathrm{diam}(\Omega)/r_0$. In (3.6), let us take b_ε small enough satisfying $C_1\left(\frac{b_\varepsilon}{a}\right)^{\frac{n\beta_2}{n-\alpha}} < \varepsilon,$ to arrive at that

$$\mu\left(\mathcal{V}\right) \le \varepsilon\mu(B(0, r_0)). \tag{3.7}$$

In the next step, let us assume that $\Omega \cap B(\xi, \varrho) \not\subset \mathcal{W}$ for some $\xi \in \Omega$ and $\varrho \in (0, r_0]$, we will show that

$$\mu\left(\mathcal{V} \cap B(\xi, \varrho)\right) \le \varepsilon\mu\left(B(\xi, \varrho)\right). \tag{3.8}$$

This assumption allows us to find $\xi_2 \in \Omega \cap B(\xi, \varrho) \setminus \mathcal{W}$ and $\xi_3 \in \mathcal{V} \cap B(\xi, \varrho)$ satisfying

$$M_\alpha(|\nabla u|^p)(\xi_2) \le \lambda \text{ and } M_\alpha(|\mathcal{G}|^p)(\xi_3) \le b_\varepsilon \lambda.$$

For this reason, by (2.3), one gets that

$$\int_{B(\xi_2, r)} |\nabla u|^p \, dx \le r^{n-\alpha}\lambda, \text{ and } \int_{B(\xi_3, r)} |\mathcal{G}|^p \, dx \le r^{n-\alpha} b_\varepsilon \lambda, \tag{3.9}$$

for every $r > 0$. For each $\zeta \in B(\xi, \varrho)$, we write

$$M_\alpha(|\nabla u|^p)(\zeta) = \max\left\{M_\alpha^\varrho(|\nabla u|^p)(\zeta); T_\alpha^\varrho(|\nabla u|^p)(\zeta)\right\}.$$

Since $B(\zeta,r) \subset B(\xi,2\varrho)$ for all $r \in (0,\varrho)$, one has

$$M_\alpha^\varrho(|\nabla u|^p)(\zeta) = \sup_{0<r<\varrho} r^{\alpha-n}\int_{B(\zeta,r)} \chi_{B(\xi,2\varrho)}|\nabla u|^p\,dx \le M_\alpha(\chi_{B(\xi,2\varrho)}|\nabla u|^p)(\zeta).$$

For every $r \ge \varrho$ there holds $B(\zeta,r) \subset B(\xi_2,3r)$, which deduces from (3.9) that

$$T_\alpha^\varrho(|\nabla u|^p)(\zeta) = \sup_{r\ge\varrho} r^{\alpha-n}\int_{B(\zeta,r)}|\nabla u|^p\,dx \le 3^n \sup_{r\ge\varrho} r^{\alpha-n}\int_{B(\xi_2,3r)}|\nabla u|^p\,dx \le 3^n\lambda.$$

Therefore, we may conclude that

$$M_\alpha(|\nabla u|^p)(\zeta) \le \max\left\{M_\alpha(\chi_{B(\xi,2\varrho)}|\nabla u|^p)(\zeta); 3^n\lambda\right\}, \quad \forall \zeta \in B(\xi,\varrho).$$

For this reason, one obtains that

$$|\mathcal{V} \cap B(\xi,\varrho)| \le \left|\left\{M_\alpha^\varrho(\chi_{B(\xi,2\varrho)}|\nabla u|^p)(\zeta) > a\lambda\right\} \cap B(\xi,\varrho)\right|, \tag{3.10}$$

provided $a > 3^n$. We now consider two cases when ξ belongs to the interior domain $B(\xi,8\varrho) \subset \Omega$ and ξ is close to the boundary $B(\xi,8\varrho) \cap \partial\Omega \neq \emptyset$. In the first case $B(\xi,8\varrho) \subset \Omega$, we set $\tilde{\xi} = \xi$ and $k_0 = 8$. Otherwise, if $B(\xi,8\varrho) \cap \partial\Omega \neq \emptyset$, then, we choose $\tilde{\xi} \in \partial\Omega$ such that $|\xi - \tilde{\xi}| = d(\xi,\partial\Omega) < 8\varrho$. In this case, we take $k_0 = 10$. For simplicity of notation, from now on, we denote

$$\tilde{B}_r = B(\tilde{\xi},k_0 r) \text{ and } \tilde{\Omega}_r = \Omega \cap B(\tilde{\xi},k_0 r), \quad \text{for every } r > 0.$$

By these choices, we always have

$$B(\xi,2\varrho) \subset \tilde{B}_\varrho. \tag{3.11}$$

Thanks to Lemma 3.1, there exists $\delta_0 \in (0,1/8)$, $\tilde{p} \in (0,p)$ and $v \in W^{1,p}(\tilde{B}_{2\varrho}) \cap W^{1,\infty}(\tilde{B}_\varrho)$ such that

$$\|\nabla v\|_{L^\infty(\tilde{B}_\varrho)}^p \le C\varrho^{-n}\left(\int_{\tilde{B}_{4\varrho}}|\nabla u|^p\,dx + \int_{\tilde{B}_{4\varrho}}|\mathcal{G}|^p\,dx\right),$$

and

$$\int_{\tilde{B}_{2\varrho}}|\nabla u - \nabla v|^p\,dx \le C\left(\delta^p\int_{\tilde{B}_{4\varrho}}|\nabla u|^p\,dx + \delta^{\tilde{p}}\int_{\tilde{B}_{4\varrho}}|\mathcal{G}|^p\,dx\right).$$

provided $(\Omega,\mathbb{A}) \in (\mathcal{H})_{\delta,r_0}$ for $\delta \in (0,\delta_0)$. Applying (3.9) with the fact that

$$\tilde{B}_\varrho \subset B(\xi_2,3k_0\varrho) \cap B(\xi_3,3k_0\varrho),$$

there holds

$$\|\nabla v\|_{L^\infty(\tilde{B}_\varrho)}^p \le C\varrho^{-\alpha}\left(1+b_\varepsilon\right)\lambda, \tag{3.12}$$

and

$$\int_{\tilde{B}_{2\varrho}}|\nabla u - \nabla v|^p\,dx \le C\varrho^{n-\alpha}\left(\delta^p + \delta^{\tilde{p}}b_\varepsilon\right)\lambda. \tag{3.13}$$

By (3.10) and (3.11), one gets that

$$\begin{aligned}\left|\mathcal{V}\cap B(\xi,\varrho)\right| &\le \left|\left\{M_{\alpha}^{\varrho}(\chi_{\tilde{B}_{\varrho}}|\nabla u|^{p}) > a\lambda\right\}\cap B(\xi,\varrho)\right| \\ &\le \left|\left\{M_{\alpha}^{\varrho}(\chi_{\tilde{B}_{\varrho}}|\nabla u-\nabla v|^{p}) > a\lambda/2\right\}\cap B(\xi,\varrho)\right| \\ &\quad + \left|\left\{M_{\alpha}^{\varrho}(\chi_{\tilde{B}_{\varrho}}|\nabla v|^{p}) > a\lambda/2\right\}\cap B(\xi,\varrho)\right|.\end{aligned} \tag{3.14}$$

We now show that the last term on the right-hand side of (3.14) vanishes for a large enough. Indeed, for every $\zeta \in B(\xi,\varrho)$, one has

$$M_{\alpha}^{\varrho}(\chi_{\tilde{B}_{\varrho}}|\nabla v|^{p})(\zeta) = \sup_{0<r<\varrho} r^{\alpha-n}\int_{B(\zeta,r)}\chi_{\tilde{B}_{\varrho}}|\nabla v|^{p}\,dx \le \varrho^{\alpha}\ \|\nabla v\|_{L^{\infty}(\tilde{B}_{\varrho})}^{p}.$$

Substituting (3.12) into this inequality, it yields that

$$M_{\alpha}^{\varrho}(\chi_{\tilde{B}_{\varrho}}|\nabla v|^{p})(\zeta) \le C_2(1+b_{\varepsilon})\lambda, \quad \text{for all } \zeta\in B(\xi,\varrho).$$

Hence, if $a/2 > C_2(1+b_{\varepsilon})$ then $\left|\left\{M_{\alpha}^{\varrho}(\chi_{\tilde{B}_{\varrho}}|\nabla v|^{p}) > a\lambda/2\right\}\cap B(\xi,\varrho)\right| = 0,$ which from (3.14) deduces that

$$\left|\mathcal{V}\cap B(\xi,\varrho)\right| \le \left|\left\{M_{\alpha}^{\varrho}(\chi_{\tilde{B}_{\varrho}}|\nabla u-\nabla v|^{p}) > a\lambda/2\right\}\cap B(\xi,\varrho)\right|.$$

Applying Lemma 2.6 and (3.13), we observe that

$$\begin{aligned}\left|\mathcal{V}\cap B(\xi,\varrho)\right| &\le C\left(\frac{2}{a\lambda}\int_{\tilde{B}_{\varrho}}|\nabla u-\nabla v|^{p}\,dx\right)^{\frac{n}{n-\alpha}} \\ &\le C\left[\frac{2}{a}\varrho^{n-\alpha}\left(\delta^{p}+\delta^{\tilde{p}}b_{\varepsilon}\right)\right]^{\frac{n}{n-\alpha}} \\ &\le C\left[\frac{\delta^{p}+\delta^{\tilde{p}}b_{\varepsilon}}{a}\right]^{\frac{n}{n-\alpha}}|B(\xi,\varrho)|.\end{aligned} \tag{3.15}$$

Combining (3.15) with the inequality (2.1), there holds

$$\mu\left(\mathcal{V}\cap B(\xi,\varrho)\right) \le C\left[\frac{\left|\mathcal{V}\cap B(\xi,\varrho)\right|}{|B(\xi,\varrho)|}\right]^{\beta_2}\mu(B(\xi,\varrho)) \le C_3\left[\frac{\delta^{p}+\delta^{\tilde{p}}b_{\varepsilon}}{a}\right]^{\frac{n\beta_2}{n-\alpha}}\mu(B(\xi,\varrho)).$$

Inequality (3.8) will be valid by choosing δ small enough such that $C_3\left[\frac{\delta^{p}+\delta^{\tilde{p}}b_{\varepsilon}}{a}\right]^{\frac{n\beta_2}{n-\alpha}} < \varepsilon.$

By statements in (3.7) and (3.8), Lemma 2.7 gives us the conclusion of (3.4), which leads to (3.3). The proof is complete. □

Theorem 3.3. *Let $\varphi\in L^1_{\mathrm{loc}}(\mathbb{R}^+;\mathbb{R}^+)$ and Φ be given as in (2.2) such that*

$$\nu_1\Phi(\lambda) \le \Phi(2\lambda) \le \nu_2\Phi(\lambda), \quad \text{for all } \lambda\ge 0, \tag{3.16}$$

for some $1<\nu_1<\nu_2$. For every $s\in(0,\infty)$ and $0<t\le\infty$, there exists $\delta=\delta(s,t,\nu_1,\nu_2,data)\in(0,1/8)$ such that if $(\Omega,\mathbb{A})\in(\mathcal{H})_{\delta,r_0}$ for some $r_0>0$, then

$$\|M_\alpha(|\nabla u|^p)\|_{\mathcal{L}^{s,t}_{\varphi,\mu}(\Omega)} \le C\ \|M_\alpha(|\mathcal{G}|^p)\|_{\mathcal{L}^{s,t}_{\varphi,\mu}(\Omega)}. \tag{3.17}$$

Proof. Thanks to Theorem 3.2, it allows us to find $a>1$, $\delta \in (0,1/8)$, and $b_\varepsilon \in (0,1)$ such that the following weighted distribution inequality

$$d_u^\mu(a\lambda) \le C_4 \varepsilon d_u^\mu(\lambda) + d_{\mathcal{G}}^\mu(b_\varepsilon \lambda) \tag{3.18}$$

holds for every $\varepsilon \in (0,\varepsilon_0)$ if $(\Omega,\mathbb{A}) \in (\mathcal{H})_{\delta,r_0}$ for some $r_0>0$. For every $s,t \in (0,\infty)$, by Definition 2.4 and the change of variable $\lambda \to a\lambda$, one gets that

$$\|M_\alpha(|\nabla u|^p)\|^t_{\mathcal{L}^{s,t}_{\varphi,\mu}(\Omega)} \le a^t s\int_0^\infty \lambda^{t-1}\left[\Phi\left(C_4\varepsilon d_u^\mu(\lambda) + d_{\mathcal{G}}^\mu(b_\varepsilon\lambda)\right)\right]^{\frac{t}{s}} d\lambda. \tag{3.19}$$

The last inequality comes from (3.18) and the fact that Φ is non-decreasing. Moreover, condition (3.16) implies that

$$\Phi\left(C_4\varepsilon d_u^\mu(\lambda) + d_{\mathcal{G}}^\mu(b_\varepsilon\lambda)\right) \le \nu_2\left[\Phi\left(C_4\varepsilon d_u^\mu(\lambda)\right) + \Phi\left(d_{\mathcal{G}}^\mu(b_\varepsilon\lambda)\right)\right]. \tag{3.20}$$

Let m and k be two natural numbers that satisfy $2^{m-1} < C_4 \le 2^m$, and $1/2 < 2^k\varepsilon \le 1$. Applying (3.16), one has

$$\Phi\left(C_4\varepsilon d_u^\mu(\lambda)\right) \le \Phi\left(2^m\varepsilon d_u^\mu(\lambda)\right) \le \nu_2^m \Phi\left(\varepsilon d_u^\mu(\lambda)\right),$$

and

$$\Phi\left(\varepsilon d_u^\mu(\lambda)\right) \le \nu_1^{-k}\Phi\left(2^k\varepsilon d_u^\mu(\lambda)\right) \le \nu_1^{-k}\Phi\left(d_u^\mu(\lambda)\right) \le \nu_1^{1+\log_2\varepsilon}\Phi\left(d_u^\mu(\lambda)\right).$$

Substituting the two estimates above into (3.20) and (3.19), we obtain that

$$\begin{aligned}\|M_\alpha(|\nabla u|^p)\|^t_{\mathcal{L}^{s,t}_{\varphi,\mu}(\Omega)} &\le \nu_2^{\frac{t}{s}} a^t s\int_0^\infty \lambda^{t-1}\left[\nu_2^m\nu_1^{1+\log_2\varepsilon}\Phi\left(d_u^\mu(\lambda)\right) + \Phi\left(d_{\mathcal{G}}^\mu(b_\varepsilon\lambda)\right)\right]^{\frac{t}{s}} d\lambda \\ &\le C\nu_1^{\frac{t}{s}(1+\log_2\varepsilon)} s\int_0^\infty \lambda^{t-1}\left[\Phi\left(d_u^\mu(\lambda)\right)\right]^{\frac{t}{s}} d\lambda \\ &\quad + C(b_\varepsilon)^{-t} s\int_0^\infty \lambda^{t-1}\left[\Phi\left(d_{\mathcal{G}}^\mu(\lambda)\right)\right]^{\frac{t}{s}} d\lambda.\end{aligned} \tag{3.21}$$

The last inequality in (3.21) can be rewritten as follows:

$$\|M_\alpha(|\nabla u|^p)\|^t_{\mathcal{L}^{s,t}_{\varphi,\mu}(\Omega)} \le C\nu_1^{\frac{t}{s}(1+\log_2\varepsilon)}\ \|M_\alpha(|\nabla u|^p)\|^t_{\mathcal{L}^{s,t}_{\varphi,\mu}(\Omega)} + Cb_\varepsilon^{-t}\ \|M_\alpha(|\mathcal{G}|^p)\|^t_{\mathcal{L}^{s,t}_{\varphi,\mu}(\Omega)},$$

which deduces to

$$\|M_\alpha(|\nabla u|^p)\|_{\mathcal{L}^{s,t}_{\varphi,\mu}(\Omega)} \le C_5\nu_1^{\frac{1}{s}(1+\log_2\varepsilon)}\ \|M_\alpha(|\nabla u|^p)\|_{\mathcal{L}^{s,t}_{\varphi,\mu}(\Omega)} + Cb_\varepsilon^{-1}\ \|M_\alpha(|\mathcal{G}|^p)\|_{\mathcal{L}^{s,t}_{\varphi,\mu}(\Omega)}. \tag{3.22}$$

Similarly, (3.22) holds for the remaining case $t=\infty$. Since $\nu_1>1$, hence $\lim_{\varepsilon\to 0^+}\nu_1^{\frac{1}{s}(1+\log_2\varepsilon)} = 0.$

For this reason, it is possible to choose ε small enough such that $C_5\nu_1^{\frac{1}{s}(1+\log_2\varepsilon)} < \frac{1}{2}$. This allows us to conclude (3.17). The proof is complete. ☐

4 Conclusion

We obtain two main results stated in Theorem 3.2 and Theorem 3.3. The first result is the weighted form of the distribution inequality. This inequality is derived from the local comparison in Lemma 3.1 by using the covering lemma. Applying this distribution inequality, we prove the gradient estimate in the generalized Lorentz space based on the rearrangement invariance characteristic of this space.

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BẤT ĐẲNG THỨC PHÂN PHỐI CHO MỘT LỚP CÁC PHƯƠNG TRÌNH ELLIPTIC TỰA TUYẾN TÍNH VỚI DỮ LIỆU HỖN HỢP

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TÓM TẮT

Bài toán về tính chính quy nghiệm cho các phương trình đạo hàm riêng đã được nhiều nhà toán học nghiên cứu trong những năm gần đây bằng nhiều phương pháp khác nhau. Với sự phát triển của giải tích điều hòa, lí thuyết Calderón-Zygmund đóng vai trò quan trọng trong việc nghiên cứu bài toán chính quy nghiệm. Trong bài báo này, chúng tôi thiết lập đánh giá dạng Calderón-Zygmund cho nghiệm yếu của một lớp phương trình elliptic tựa tuyến tính với dữ liệu hỗn hợp trong không gian Lorentz tổng quát. Nghiên cứu của chúng tôi là một dạng mở rộng liên quan đến không gian hàm đối với một số đánh giá gradient trong một số bài báo mới đây. Kết quả này một lần nữa khẳng định tính hiệu quả của phương pháp sử dụng bất đẳng thức phân phối trên các tập mức đối với bài toán chính quy nghiệm cho phương trình đạo hàm riêng.

Từ khóa: đánh giá dạng Calderón-Zygmund; bất đẳng thức hàm phân phối; toán tử cực đại cấp phân số; không gian Lorentz tổng quát; phương trình elliptic tựa tuyến tính