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Research Article

EXISTENCE OF NONTRIVIAL NONNEGATIVE WEAK SOLUTIONS FOR A CLASS OF LOGISTIC-TYPE SYSTEMS

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ABSTRACT

We consider the following logistic system

$$\begin{cases} -\Delta_p u = \lambda f_1(x,u,v) - g_1(x,u) & \text{in } \Omega, \\ -\Delta_p v = \lambda f_2(x,v,u) - g_2(x,v) & \text{in } \Omega, \\ u = 0, v = 0 & \text{on } \partial\Omega. \end{cases}$$

Assuming that the nonlinearities f_i and g_i satisfy certain growth conditions. We use the fixed point index theory and monotone minorants techniques to prove the existence of solutions for the system. This extends some known results.

Keywords: fixed point index; logistic system; *(p-1)* – sublinear

1. Introduction

In this paper, we study the system

$$\begin{cases} -\Delta_p u = \lambda f_1(x,u,v) - g_1(x,u) & in\ \Omega, \\ -\Delta_p v = \lambda f_2(x,v,u) - g_2(x,v) & in\ \Omega, \\ u = v = 0 & on\ \partial\Omega, \end{cases} \tag{1}$$

where u,v are non-trivial, non-negative unknown functions, $\Omega \subset \mathbb{R}^N (N \geq 2)$ represents a bounded domain with a smooth boundary $\partial\Omega$, $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ denotes the p-Laplacian with $1 < p < N$, $\lambda > 0$ is a real parameter, and $f_i, g_i, i = 1,2$ are appropriately chosen functions.

Our objective is to identify a solution (u,v) that satisfies the condition of $u,v \geq \theta, u \neq \theta, v \neq \theta$.

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Based on the growth rate of the functions concerning their second variable, we classify the system of equations (1) into three categories: $(p-1)$-sublinear, $(p-1)$-linear, and $(p-1)$- superlinear.

The system (1) extends the symbiotic model that describes the coexistence of two species within the same habitat. Specific forms of this model have been studied (Delgado et al., 2000; Yang & Wang, 2007). According to the aforementioned classification, Delgado et al., (2000) and Yang and Wang (2007) focused on the $(p-1)$-linear case. In a previous work, we investigated problem (1) in the $(p-1)$- sublinear case (Nguyen & Bui, 2017). In this paper, we explore the $(p-1)$-linear, and $(p-1)$- superlinear growth cases for the second variable in the functions $f_i, i=1,2$.

Our research approach aligns with earlier studies. First, we reduce problem (1) to a fixed-point problem. Then, by determining the degree of the solution operator, we establish the existence of solutions for the system.

2. Eigenfunction and eigenvalue

Proposition 2.1. *Assume that λ_0 is the principal eigenvalue of the problem*

$$\begin{cases} -\Delta_p u = \lambda m(x)|u|^{p-2}u \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{2}$$

and $v_0 \in C_0^\infty(\bar{\Omega}), v_0 \geq \theta, v_0 \not\equiv \theta$.

Then, the problem

$$\begin{cases} -\Delta_p u = \lambda m(x)|u|^{p-2}u + v_0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega, \end{cases} \tag{3}$$

Has no non- negative solution $u \geq \theta$ if $\lambda > \lambda_0$.

Proof. By contradiction, we assume that $w \in K$ is a solution to the problem (3). Since $\lambda > \lambda_0$ and $v_0 \geq \theta$ then w is an upper solution of the following problem:

$$\begin{cases} -\Delta_p u = \lambda_0 m(x)|u|^{p-2}u + v_0 \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{cases} \tag{4}$$

It is easy to see that $u = \theta$ is a lower solution of (4). Therefore, problem (4) has a solution u (Drabek & Hernandez, 2001) satisfying $\theta \leq u \leq w$. According to the theorem on nonlinear regularity (Papageorgiou et al., 2009), we deduce that $u \in C_0^1(\bar{\Omega})$. According to Vázquez's strong maximum principle (Vázquez, 1984), we have $u \in \operatorname{int}(C_+)$. Let u_0 be the eigenfunction corresponding to the principal eigenvalue λ_0 of the problem (2). Applying the Díaz-Saa-Brezis inequality (Diáz et al., 1987), we obtain

$$\frac{v_0}{u^{p-1}}\left(u^{p-1}-(tu_0)^{p-1}\right)=\left[\frac{-\Delta_p u}{u^{p-1}}-\frac{-\Delta_p (tu_0)}{(tu_0)^{p-1}}\right]\left(u^{p-1}-(tu_0)^{p-1}\right)\geq 0.$$

Letting t tend to infinity, we obtain $\frac{v_0}{u^{p-1}}=\theta$, which contradicts the assumption about v_0.

3. A reduction to the fixed point equations

Let $\Omega\subset\mathbb{R}^N$ be a bounded domain with a smooth boundary $1<p<N$. We denote the norms in the spaces $W_0^{1,p}(\Omega)$ and $L^t(\Omega)$ by $\|.\|$ and $\|.\|_t$ respectively. In these spaces, we consider the order cone of nonnegative functions.

In this paper, we make the following assumptions on the functions f_i and g_i, $i=1,2$.

- *(g1)* $g_i(x,0)=0$, *and* $g_i(x,u)$ *is an increasing function with respect to* u *for almost every* $x\in\Omega$.
- *(g2) There exist constants* $a_i>0$, $0<\beta<p^*-1$, *and functions* $b_i\in L^{(\beta+1)'}(\Omega)$ *such that*

$$|g_i(x,u)|\leq a_i|u|^\beta+b_i(x),\quad \forall(x,u)\in\Omega\times\mathbb{R}.$$

- *(f)* $|f_i(x,u,v)|\leqslant m_i(x)|u|^\alpha+n_i(x)|v|^\gamma$, *where* $\alpha,\gamma<p^*-1$ *and* $m_i\in L^q(\Omega),n_i\in L^r(\Omega)$ *with* $q>(\frac{p^*}{1+\alpha})', r>(\frac{p^*}{1+\gamma})'$.

Then, Nguyen and Bui (2017) established that $(u,v)\in W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ is a solution to problem (1) if and only if $(u,v)=Po\lambda N(u,v)$, where $PoN:=(P_1oN_{f_1},P_2oN_{f_2})$ and $N_f(u,v)(x)=f(x,u(x),v(x)),\quad \forall u,v\in W_0^{1,p}(\Omega).$

4. The main results

Theorem 4.1. **(**The case $(p-1)$ – linear) *Suppose that the Caratheodory functions* $g_i:\Omega\times\mathbb{R}^+\to\mathbb{R}^+$, $f_i:\Omega\times\mathbb{R}^+\times\mathbb{R}^+\to\mathbb{R}^+$, $i=1,2$ *satisfy the condition (g1) and*

(H1) (a) $a_i z^\beta-b_i(x)\leq g_i(x,z)\leq e_i z^\beta+b_i(x)$ *where* $a_i,e_i>0$ *and* $p-1<\beta<p^*-1$,

$$b_i(x)\in L^s(\Omega), s=max\left\{(\beta+1)',\frac{N}{p}\right\}, i=1,2;$$

(b) $\lim_{z\to 0^+}\frac{g_i(x,z)}{z^{p-1}}=0$ *uniformly in* Ω, $i=1,2$.

(H2)(a) $0\leq f_i(x,z,t)\leq m_i(x)z^{p-1}+n_i(x)t^{p-1},\forall(x,z,t)\in\Omega\times\mathbb{R}^+\times\mathbb{R}^+$ *where*

$$m_i\in L^q(\Omega), q>\left(\frac{p^*}{p}\right)',\ n_i\in L^r(\Omega), r>\left(\frac{(\beta+1)p}{(p-1)p+1+\beta}\right)', i=1,2.$$

(b) $f_i(x,z,t)>0,\forall(x,z,t)\in\Omega\times\mathbb{R}^+\times\mathbb{R}^+,(z,t)\neq(0,0).$

(H3) There exist the functions $p_i \geq \theta, p_i \neq \theta, p_i \in L^s(\Omega)$ *where* $s > \dfrac{N}{p}$ *and the constant numbers* e_i *such that for every consequence satisfying* $\xi_n \to 0^+,\ z_n \to z,\quad t_n \to t$ *then*

$$\lim_{n\to\infty} \frac{f_i(x, \xi_n z_n, \xi_n t_n)}{\xi_n^{p-1}} = p_i(x) z^{p-1} + e_i t^{p-1},\ i = 1,2.$$

Let $\lambda_i, i = 1,2$ *be the eigenvalues corresponding to the following problems:*

$$-\Delta_p u = \lambda p_i(x) |u|^{p-2} u \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega.$$

Then, for every $\lambda > \max\{\lambda_1, \lambda_2\}$, *problem (1) has at least a solution* (u,v) *such that* $u, v \geq \theta, u \neq \theta, v \neq \theta.$

Proof. We will still denote $N(u,v) = (\lambda N_{f_1}(u,v), \lambda N_{f_2}(u,v))$ and still split the proof into three steps.

Step 1. We will show that there exists a sufficiently large positive number R such that

$$(u,v) \neq P[tN(u,v)], \forall t \in [0,1], \forall u, v \geq \theta, \|(u,v)\| = R.$$

Assume the contrary that there exist sequences $\{t_n\} \subset [0,1]$, and $u_n, v_n \geqslant \theta,\ \|(u_n, v_n)\| \to \infty$ such that $(u_n, v_n) = P[t_n N(u_n, v_n)]$, or equivalently, one has

$$\begin{cases} \langle Au_n, \varphi\rangle + \int_\Omega g_1(x, u_n)\varphi = \int_\Omega t_n f_1(x, u_n, v_n)\varphi \\ \langle Av_n, \phi\rangle + \int_\Omega g_2(x, v_n)\phi = \int_\Omega t_n f_2(x, v_n, u_n)\phi \end{cases} \quad \forall \varphi, \phi \in W_0^{1,p}(\Omega). \tag{2}$$

Choosing $\varphi = u_n, \phi = v_n$ in (7) and using $(H2)(a)$, we obtain

$$\begin{cases} \|u_n\|^p + \int_\Omega g_1(x, u_n)u_n \leqslant \int_\Omega m_1(x) u_n^p + \int_\Omega n_1(x) v_n^{p-1} u_n, \\ \|v_n\|^p + \int_\Omega g_2(x, v_n)v_n \leqslant \int_\Omega m_2(x) v_n^p + \int_\Omega n_2(x) u_n^{p-1} v_n. \end{cases} \tag{3}$$

Adding sides by sides of the inequalities in (3), we have

$$C.\ \|(u_n, v_n)\|^p \leq \int_\Omega m_1(x) u_n^{1+\alpha} + \int_\Omega m_2(x) v_n^p + \int_\Omega n_1(x) v_n^{p-1} u_n + \int_\Omega n_2(x) u_n^{p-1} v_n. \tag{4}$$

By Holder's inequality, Young's inequality, and some simple computations, we obtain

$$\begin{aligned} \|(u_n, v_n)\|^p + C\|(u_n, v_n)\|_{\beta+1}^{\beta+1} \leq C\left(\|m_1\|_q + \|m_2\|_q\right)\|(u_n, v_n)\|_{pq'}^p + \\ + \int_\Omega [n_1(x) v_n^{p-1} u_n + n_2(x) u_n^{p-1} v_n] + \int_\Omega [b_1(x) u_n + b_2(x) v_n] \end{aligned} \tag{5}$$

Applying the Hölder and Young's inequalities, we get

$$\int_\Omega [n_1(x)v_n^{p-1}u_n + n_2(x)u_n^{p-1}v_n] \le C\left[\|n_1\|_r + \|n_2\|_r\right]\left[\varepsilon\|(u_n,v_n)\|_p^{\frac{p}{r'}} + C(\varepsilon)\|(u_n,v_n)\|_t^{\frac{t}{r'}}\right] \quad (6)$$

where $t = (p-1)r'\left(\dfrac{p}{r'}\right)' = \dfrac{(p-1)pr'}{p-r'} < 1+\beta.$

From (5), (6), we have

$$\|(u_n,v_n)\|^p + \|(u_n,v_n)\|_{\beta+1}^{\beta+1} \le C\left(\|(u_n,v_n)\|_{pq'}^p + \|(u_n,v_n)\|_t^{\frac{t}{r'}} + 1\right). \quad (7)$$

Since $r > \left(\dfrac{(\beta+1)p}{(p-1)p+1+\beta}\right)'$ then $t < \beta+1$. Therefore, (6) follows

$$\|(u_n,v_n)\|^p + \|(u_n,v_n)\|_{\beta+1}^{\beta+1} \le C\|(u_n,v_n)\|_{pq'}^p. \quad (8)$$

If $pq' \le \beta+1$ then $\|(u_n,v_n)\|^p + \|(u_n,v_n)\|_{\beta+1}^{\beta+1} \le C\|(u_n,v_n)\|_{\beta+1}^p$. (9)

From (9) we get $\|(u_n,v_n)\|_{\beta+1} \to \infty$. This contradicts $p < \beta+1$.

If $pq' > \beta+1$, it follows from the hypothesis $(H4)$, we obtain $pq' < p^*$. Applying the interpolation inequality, we obtain

$$\begin{aligned}\|(u_n,v_n)\|_{pq'} = \|u_n\|_{pq'} + \|v_n\|_{pq'} &\le C\left(\|u_n\|_{p^*}^\theta.\|u_n\|_{\beta+1}^{1-\theta} + \|v_n\|_{p^*}^\theta.\|v_n\|_{\beta+1}^{1-\theta}\right)\\ &\le C\left(\|u_n\|_{p^*}^\theta + \|v_n\|_{p^*}^{1-\theta}\right)\left(\|u_n\|_{\beta+1}^{1-\theta}.\|v_n\|_{\beta+1}^{1-\theta}\right)\\ &\le C\|(u_n,v_n)\|_{p^*}^\theta.\|(u_n,v_n)\|_{\beta+1}^{1-\theta}\\ &\le C\|(u_n,v_n)\|_{p^*}^\theta.\|(u_n,v_n)\|_{\beta+1}^{1-\theta}\end{aligned} \quad (10)$$

where $\theta \in (0;1)$ is defined by

$$\frac{1}{\beta+1} - \frac{1}{pq'} = \theta\left(\frac{1}{\beta+1} - \frac{1}{p^*}\right).$$

It follows from (9), we get

$$\|(u_n,v_n)\| \le C.\|(u_n,v_n)\|_{pq'} \text{ and } \|(u_n,v_n)\|_{\beta+1} \le \|(u_n,v_n)\|_{pq'}^{\frac{p}{\beta+1}},$$

Combined with (10), we obtain $\|u_n\|_{pq'} \le C\|u_n\|_{pq'}^{\theta+(1-\theta)\frac{p}{\beta+1}}$.

Note that $p < 1+\beta$ then $\theta + (1-\theta)\dfrac{p}{1+\beta} < 1$. Therefore, in the above inequality, we have a contradiction with $\|(u_n,v_n)\|_{pq'} \to \infty$.

Step 2. Choosing arbitrarily, $\varphi_1,\varphi_2 \in C_0^\infty\left(\overline{\Omega}\right)\setminus\{\theta\}$, we prove that there exists a small enough positive number $r>0$ so that

$$(u,v)\neq P\left[N(u,v)+t(\varphi_1,\varphi_2)\right],\forall t>0,\forall u\geq 0,\|u\|=r.$$

Assume the above statement is not true, then we can find sequences $t_n>0,u_n,v_n\geq\theta,\|(u_n,v_n)\|\to 0$ such that $(u_n,v_n)=P\left[N(u_n,v_n)+t_n(\varphi_1,\varphi_2)\right]$. It means that

$$\langle Au_n,\varphi\rangle=\int_\Omega\left[\lambda f_1(x,n_n,v_n)-g_1(x,u_n)+t_n\varphi_1\right]\varphi,\forall\varphi\in W_0^{1,p}(\Omega) \quad (11)$$

and

$$\langle Av_n,\varphi\rangle=\int_\Omega\left[\lambda f_2(x,v_n,v_n)-g_2(x,v_n)+t_n\varphi_1\right]\varphi,\forall\varphi\in W_0^{1,p}(\Omega) \quad (12)$$

Divide both sides of equation (11) by $\|(u_n,v_n)\|^{p-1}$, we have

$$\langle Az_n,\varphi\rangle=\int_\Omega\left[\lambda f_1(x,u_n,v_n)-g_1(x,u_n)+t_n\varphi_1\right]\frac{\varphi}{\|(u_n,v_n)\|^{p-1}}, \quad (13)$$

where $z_n=\dfrac{u_n}{\|(u_n,v_n)\|}$.

Since $\{z_n\}$ is a bounded sequence in a reflexive space $W_0^{1,p}(\Omega)$, we can assume without loss of generality that $z_n\to z$ weakly in $W_0^{1,p}(\Omega)$ and $z_n\to z$ in $L^{\delta'}(\Omega)$ (note that $\delta'<p^*$).

We prove the following statements:

The sequence $\left\{\dfrac{t_n}{\|(u_n,v_n)\|^{p-1}}\right\}$ is bounded; (14)

The sequence $\left\{\dfrac{f_1(x,u_n,v_n)}{\|(u_n,v_n)\|^{p-1}}\right\}$ is bounded in $L^\delta(\Omega)$; (15)

$\lim\int_\Omega\dfrac{g_1(x,u_n)\varphi}{\|(u_n,v_n)\|^{p-1}}=0$ uniformly with respect to φ in all bounded subsets of $W_0^{1,p}(\Omega)$. (16)

Indeed, statement (15) follows from

$$0\leq\frac{f_1(x,u_n,v_n)}{\|(u_n,v_n)\|^{p-1}}\leq m_1(x)z_n^{p-1}+n_1(x)w_n^{p-1}, \quad (17)$$

where $w_n=\dfrac{v_n}{\|(u_n,v_n)\|^{p-1}}$, and the map $(z,w)\mapsto m_1(x)z^{p-1}+n_1(x)w^{p-1}$ transforms a bounded set in $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ into a bounded set in $L^\delta(\Omega)$ ().

In order to prove (21), let $\varepsilon>0$ be a fixed positive number, choosing $\delta>0$ so that

$g_1(x,u) < \varepsilon u^{p-1},$ where $u < \delta, x \in \Omega.$

Pose that $\Omega_n = \{x \in \Omega : u_n(x) \geq \delta\}$, we have

$$\int_{\Omega\setminus\Omega_n} \frac{g_1(x,u_n)|\varphi|}{\|(u_n,v_n)\|^{p-1}} \leq \varepsilon \int_{\Omega\setminus\Omega_n} z_n^{p-1}|\varphi| \leq \varepsilon \int_{\Omega} z_n^{p-1}|\varphi| \leq C\varepsilon\|\varphi\| \quad (18)$$

and

$$\int_{\Omega_n} \frac{g_1(x,u_n)|\varphi|}{\|(u_n,v_n)\|^{p-1}} \leq \int_{\Omega_n} \frac{e_1 u_n^{\beta}|\varphi|}{\|(u_n,v_n)\|^{p-1}} + \int_{\Omega_n} \frac{b_1|\varphi|}{\|(u_n,v_n)\|^{p-1}}$$

$$\leq e_i \|(u_n,v_n)\|^{\beta-p+1} \int_{\Omega} z_n^{\beta}|\varphi| + \frac{1}{\delta^{p-1}} \int_{\Omega_n} b_1|\varphi| z_n^{p-1} \quad (19)$$

$$\leq e_i \|(u_n,v_n)\|^{\beta-p+1}|\varphi| + C\left(\int_{\Omega_n} b_1^{\frac{N}{p}}\right)^{\frac{p}{N}} \frac{\|\varphi\|}{\delta^{p-1}}.$$

In the above inequalities, we have used $\varphi \in L^{p^*}(\Omega), z_n^{p-1} \in L^{\frac{p^*}{p-1}}(\Omega), b_1 \in L^{\frac{N}{p}}(\Omega)$ and $\frac{1}{p^*} + \frac{p-1}{p^*} + \frac{p}{N} = 1.$ On the other hand, from

$$\delta^{p^*}.meas(\Omega_n) \leq \int_{\Omega_n} u_n^{p^*} \leq C\|u_n\|^{p^*},$$

Implies that $meas(\Omega_n) \to 0.$ Combining (18) and (19) with the absolute continuity of the integral, we have (16). Hence, from (11), it is implied that the sequence $\left\{\frac{t_n}{\|(u_n,v_n)\|^{p-1}}\right\}$ is bounded, and we can assume that it converges to some t_0 .

Chossing $\varphi = z_n - z$ in (11) and applying (15), (16), and $z_n - z \to \theta$ in $L^{\delta'}(\Omega)$, we have $\lim\langle Az_n, z_n - z\rangle = 0.$ Hence, $z_n - z$ in $W_0^{1,p}(\Omega).$

By similar reasoning, we also have $w_n = \frac{v_n}{\|(u_n,v_n)\|} \to w$ in $W_0^{1,p}(\Omega).$

Since $\|z\| + \|w\| = 1$, then $(z,w) \neq (\theta,\theta).$ Assume that $z \neq \theta.$

Because of $z_n \to z, w_n \to w$ in $L^{p^*}(\Omega)$, we may assume that $z_n \to z, w_n \to w$ a.e. in Ω and $|z_n| \leq z_0 \in L^{p^*}, |w_n| \leq w_0 \in L^{p^*}$. Then,

$$\lim_{n\to\infty}\frac{\lambda f_1\left(x,u_n,v_n\right)\varphi}{\left\|\left(u_n,v_n\right)\right\|^{p-1}}=\lim_{n\to\infty}\frac{\lambda f_1\left(x,\left\|\left(u_n,v_n\right)\right\|z_n,\left\|\left(u_n,v_n\right)\right\|w_n\right)\varphi}{\left\|\left(u_n,v_n\right)\right\|^{p-1}}$$

$$=\lambda\left(p_1(x)z^{p-1}+e_1w^{p-1}\right)\varphi,\quad \text{a.e. in } \Omega,$$

and

$$\frac{\lambda f_1\left(x,u_n,v_n\right)}{\left\|\left(u_n,v_n\right)\right\|^{p-1}}\le\lambda\left(m_1(x)z_n^{p-1}+n_1(x)w_n^{p-1}\right)$$

$$\le\lambda\left(m_1(x)z_0^{p-1}+n_1(x)v_0^{\gamma}\right)\in L^{\delta}\subset L^{\left(p^*\right)'}.$$

This implies that

$$\lim_{n\to\infty}\frac{\lambda f_1\left(x,u_n,v_n\right)\varphi}{\left\|\left(u_n,v_n\right)\right\|^{p-1}}=\int_\Omega\lambda\left(p_1(x)z^{p-1}+e_1w^{p-1}\right)\varphi, \tag{20}$$

It follows from (11), (15), (16), and (19), we have

$$\left\langle Az,\varphi\right\rangle=\int_\Omega\left(\lambda p_1(x)z^{p-1}+e_1w^{p-1}+t_0\varphi_1\right)\varphi,\forall\varphi\in W_0^{1,p}\left(\Omega\right). \tag{21}$$

If $e_1w^{p-1}+t_0\varphi_1=\theta$, we have a contradiction with $\lambda>\lambda_1$ and $z\ge\theta,z\ne\theta.$ Else, $e_1w^{p-1}+t_0\varphi_1\ne\theta$, assertion (21) contradicts Proposition 2.1.

Step 3. From Steps 1 and 2, we get

$i(PoN,B((\theta,\theta),R),K)=1,$ for large R,

and

$i(PoN,B((\theta,\theta),r),K)=0,$ as r is small.

Therefore, there exists $(u,v)\ge(\theta,\theta)$ such that $r\le\ \|(u,v)\|\ \le R$ and $(u,v)=PoN(u,v)$. This means that problem (1) has a non-negative solution.

We further prove $u\ne\theta$ và $v\ne\theta.$ Suppose that the solution $\left(u,v\right)$ of problem (1) has $u=\theta.$ By conditions $g\left(x,0\right)=0$ and (H4) (b), we have

$\theta=\Delta_p u+g\left(x,u\right)=\lambda f_1\left(x,0,v\right)\ne\theta.$

The above contradiction implies the statement to be proved.

Theorem 3.2. (The (p – 1) – super-linear) *Assume that the following conditions hold.*

(H4) The functions $g_i:\overline{\Omega}\times\mathbb{R}^+\to\mathbb{R}^+, i=1,2$ *are continuous, satisfy the condition* $\left(g_1\right)$ *and the following conditions:*

1. *There exists a number* $\beta\in\left(p-1,p^*-1\right)$ *such that* $\lim_{t\to\infty}\frac{g_i(x,t)}{t^\beta}=a_i>0$ *uniformly with* $x\in\overline{\Omega}$..

2. For each $\zeta_i > 0$, *there exists* $\sigma_i > 0$ *such that* $t \mapsto \sigma_i t^\beta - g_i(x,t)$ *is increasing on* $[0,\zeta_1], i = 1,2.$

3. $\lim_{t\to 0^+} \frac{g_i(x,t)}{t^{p-1}} = l_i < \infty$ *uniformly with* $x \in \overline{\Omega}$, *and the functions* $t \mapsto \frac{g_i(x,t)}{u^{p-1}}$ *are increasing with a.e.* $x \in \Omega$.

(H5) The functions $f_i : \overline{\Omega} \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ *are continuous and*

$$m_{1i}u^\alpha + d_i v^\eta < f_i(x,u,v) \le m_{2i}u^\alpha + c_i u^\gamma,$$

where $c_i, d_i, m_{1i}, m_{2i} > 0,\ p-1 < \gamma < p\frac{\beta}{\beta+1},\ p-1 < \alpha < \beta, \eta > 0, i = 1,2.$

Then, there exists $\overline{\lambda}$ *such that if* $\lambda > \overline{\lambda}$ *then problem (1) admits at least two non-negative, non–trivial solutions.*

Proof. In space $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$ with the usual norm $\|(u,v)\|_{C^1} = \|u\|_{C^1} + \|v\|_{C^1}$, consider the cone

$$K = \left\{(u,v) \in C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega}) : u(x), v(x) \ge 0\right\},$$

and the interior of the cone

$$\operatorname{int} K = \left\{(u,v) \in C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega}) : u(x), v(x) > 0 \text{ in } \Omega, \text{ and } \frac{\partial u}{\partial n}(x) < 0, \frac{\partial v}{\partial n}(x) < 0 \text{ on } \partial\Omega\right\}.$$

Then, the operator $P \circ N$ is compact from $C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$ within itself.

From assumption (H4) (i), it follows that there exist positive numbers $a_{1i}, a_{2i}, b_i, i = 1,2$ such that

$$a_{1i}t^\beta - b_i \le g_i(x,t) \le a_{2i}t^\beta + b_i, \forall (x,t) \in \Omega \times \mathbb{R}^+, i = 1,2.$$

Step 1. Fix $(u_0, v_0) \in K \setminus \{(\theta,\theta)\}$. We prove that for sufficiently large R, we have

$$P[\lambda N(u,v)] - (u,v) \ne t(u - u_0, v - v_0), \forall t > 0, u, v \ge \theta, \|(u,v)\|_{C^1} = R. \quad (22)$$

Assume the contrary $P[\lambda N(u_n, v_n)] - (u_n, v_n) = t_n(u_n - u_0, v_n - v_0)$ satisfies the sequences $t > 0, u_n, v_n \ge \theta, \|(u_n, v_n)\|_{C^1} \to \infty$.

We consider the following three cases.

Case 1. $\|u_n\|_{C^1} \to \infty, \{\|v_n\|_{C^1}\}$ is bounded. Let $z_n = z_n = u_n + t_n(u_n - u_0)$ we obtain

$$z_n = P_1(\lambda N_1(u_n, v_n)) \quad \text{and} \quad u_n = \frac{1}{1+t_n} z_n + \frac{t_n}{t_n + 1} u_0. \quad (23)$$

From (22), we deduce $\|z_n\|_{C^1} \to \infty$. Based on Lieberman's results on regularity (Lieberman, 1988), then $\|z_n\|_C = \max_{\bar{\Omega}} |z_n(x)| \to \infty$ and $\|z_n\| \to \infty$.

From (22) and the definition of the operator P_1, then

$$\|z_n\|^p = \int_\Omega g_1(x, z_n) z_n = \lambda \int_\Omega f_1(x, u_n, v_n) z_n .$$

This implies that

$$\|z_n\|^p = a_{11} \|z_n\| \frac{\beta+1}{\beta+1} \le \lambda \left(\int_\Omega m_{21} u_n^\alpha z_n + \int_\Omega C_1 v_n^\gamma z_n \right) + b_1 \int_\Omega z_n$$

$$\le C \|z_n + u_0\|_{\alpha+1}^{\alpha+1} + C(\varepsilon) \|v_n\|_{\gamma(\beta+1)'}^{\gamma(\beta+1)'} + \varepsilon \|z_n\|_{1+\beta}^{1+\beta} + C . \tag{24}$$

Since $\{\|v_n\|\}_{C^1}$ is bounded, then $\left\{\|v_n\|_{\gamma(\beta+1)'}^{\gamma(\beta+1)'}\right\}$ is also bounded. From (24), we get

$$\|z_n\|^p + \|z_n\|_{\beta+1}^{\beta+1} \le C\left(\|z_n\|_{\alpha+1}^{\alpha+1} + 1\right) \le C\left(\|z_n\|_{\beta+1}^{\alpha+1} + 1\right).$$

The final inequality leads to $\|z_n\|_{\beta+1} \to \infty$, and this, also due to the final inequality, contradicts $\alpha < \beta$.

Case 2. $\{\|(u_n)\|\}_{C^1}$ is bounded, and $\|(v_n)\|_{C^1} \to \infty$. Applying the reasoning similar to **Case 1**, we obtain a contradiction.

Case 3. $\|u_n\|_{C^1} \to \infty$, $\|v_n\|_{C^1} \to \infty$. Let $z_n = u_n + t_n(u_n - u_0)$, $w_n = v_n + t_n(v_n - v_0)$. By reasoning similar to **Case 1**, we obtain $\|z_n\| \to \infty$ and $\|w_n\| \to \infty$.

Using simple calculations and reasoning similar to **Case 1**, it is easy to see that

$$\|z_n\|^p + \|w_n\|^p + a\left(\|z_n\|_{\beta+1}^{\beta+1} + \|w_n\|_{\beta+1}^{\beta+1}\right) \le \left(\|z_n + u_0\|_{\alpha+1}^{\alpha+1} + \|w_n + v_0\|_{\alpha+1}^{\alpha+1}\right) + C(\varepsilon)\left(\|u_n\|_{\gamma(\beta+1)'}^{\gamma(\beta+1)'} + \|v_n\|_{\gamma(\beta+1)'}^{\gamma(\beta+1)'}\right) + \varepsilon\left(\|z_n\|_{\beta+1}^{\beta+1} + \|w_n\|_{\beta+1}^{\beta+1}\right) + C. \tag{25}$$

We again have $\|u_n\|_{\gamma(\beta+1)'}^{\gamma(\beta+1)'} \le C\|u_n\|_{p^*}^{\gamma(\beta+1)'} \le C\|u_n\|^{\gamma(\beta+1)'} \le C\|z_n\|^{\gamma(\beta+1)'}$ for sufficiently large n. Similarly, we also have $\|v_n\|_{\gamma(\beta+1)'}^{\gamma(\beta+1)'} \le C\|w_n\|^{\gamma(\beta+1)'}$ for sufficiently large n. From this and from (30), we deduce

$$\|(z_n, w_n)\|^p + \|(z_n, w_n)\|_{\beta+1}^{\beta+1} \le C\left(\|(z_n, w_n)\|_{\alpha+1}^{\alpha+1} + \|(z_n, w_n)\|^{\gamma(\beta+1)'} + 1\right)$$

$$\le C\left(\|(z_n, w_n)\|_{\beta+1}^{\alpha+1} + \|(z_n, w_n)\|^{\gamma(\beta+1)'} + 1\right). \tag{26}$$

The final inequality leads to $\|(z_n, w_n)\|_{\beta+1} \to \infty$. This contradicts $\alpha < \beta$ and $\gamma(\beta+1)' < p$.

Step 2. We prove that for sufficiently small r, we have

$$(u,v) \neq P\left[t\lambda N(u,v)\right], \forall t \in [0,1], \forall u,v \geq 0, \left\|(u,v)\right\|_{C^1} = r.$$

Assume the contrary, then $(u,v) = P\left[t\lambda N(u,v)\right]$ holds for some sequences $t_n \in [0,1], u_n, v_n \geq 0, \left\|(u_n,v_n)\right\|_{C_1} = \|u_n\|_{C_1} + \|v_n\|_{C_1} \to 0$. Then, we have $\|u_n\| \to 0$ and

$$\|u_n\|^p \leq m\|u_n\|_{1+\alpha}^{1+\alpha} + C\int_\Omega v_n^\gamma u_n. \tag{27}$$

Since $\left\{\|v_n\|_{C_1}\right\}$ is bounded, we have

$$\int_\Omega v_n^\gamma u_n \leq C\|u_n\|_{\beta+1}^{\beta+1}.$$

On the other hand, since $1+\alpha < 1+\beta < p^*$ then from (27), we get

$$\|u_n\|^p \leq C\left(\|u_n\|^{1+\alpha} + \|u_n\|^{1+\beta}\right).$$

This is not possible because of $1+\alpha > p, 1+\beta > p$ and $\|u_n\| \to 0$.

Step 3. We consider the following problem as a special case of problem (1)

$$\begin{cases} -\Delta_p u = \lambda u^\alpha - g_1(x,u) & in\ \Omega, \\ -\Delta_p v = \lambda v^\alpha - g_2(x,u) & in\ \Omega, \\ u = v = 0 & on\ \partial\Omega. \end{cases} \tag{28}$$

With the functions g_1, g_2 and the number α as above. From assumption (H4), we can prove the existence of λ_* such that if $\lambda > \lambda_*$, then problem (28) admits a solution $(u_\lambda, v_\lambda) \in int\, K$ (Iannizzotto & Papageorgiou, 2011). Assume that $\lambda > \bar{\lambda} := \frac{\lambda_* \cdot 2^\alpha}{m_0}$ where $m_0 = \min\{m_{1i}, i = 1,2\}$. Since $\frac{\lambda m_0}{2^\alpha} > \lambda_*$, there exists $(u_0, v_0) \in \operatorname{int} K$ such that

$$\begin{cases} -\Delta_p u_0 = \dfrac{\lambda m_0}{2^\alpha} u_0^\alpha - g_1(x,u_0), \\ -\Delta_p v_0 = \dfrac{\lambda m_0}{2^\alpha} v_0^\alpha - g_2(x,u_0). \end{cases}$$

Define the function $\varphi : C_0^1(\bar{\Omega}) \times C_0^1(\bar{\Omega}) \to \mathbb{R}$ as follows

$$\varphi(u,v) = \sup\left\{t \in \mathbb{R} : (u,v) \geq t(u_0,v_0)\right\}.$$

The function φ defined above is continuous and convex (Nguyen et al., 2016).

Choose $R > \left\|(y_0,v_0)\right\|_{C^1}$ so that (27) is satisfied and set

$$G = \left\{(u,v) \in K, \left\|(u,v)\right\|_{C^1} < R, \varphi(u,v) > \frac{1}{2}\right\}.$$

We will prove that $i\left(P\left(\lambda N\right),G,K\right)=1$ by applying Proposition 2.2 (Nguyen et al., 2016). It is easy to see that it is sufficient to check if $\varphi\left(u,v\right)=\frac{1}{2}$ then $\varphi\left[P\left[\lambda N\left(u,v\right)\right]\right]>\frac{1}{2}$. Indeed, let $\left(z,w\right)=P\left[\lambda N\left(u,v\right)\right]$, then

$$-\Delta_p z+g_1\left(x,z\right)=\lambda f_1\left(x,u,v\right)\geq\lambda m_{11}u^\alpha\geq\lambda m_0 u^\alpha$$

$$\geq\frac{\lambda m_0}{2^\alpha}u_0^\alpha=-\Delta_p u_0+g_1\left(x,u_0\right)$$

and

$$-\Delta_p w+g_2\left(x,w\right)=\lambda f_2\left(x,u,v\right)\geq\lambda m_{12}v^\alpha\geq\lambda m_0 v^\alpha$$

$$\geq\frac{\lambda m_0}{2^\alpha}v_0^\alpha=-\Delta_p v_0+g_2\left(x,u_0\right).$$

Hence, we get $z\geq u_0, w\geq v_0$ or $\varphi\left(z,w\right)\geq 1$.

On the other hand, by Steps 1 and 2, we have

$i\left(P\circ\left(\lambda N\right),B\left(\theta,R\right),K\right)=1$ for sufficiently large R,

$i\left(P\circ\left(\lambda N\right),B\left(\theta,r\right),K\right)=1$ for sufficiently small r.

From this, we get $B\left(\theta,r\right)\cup G\subset B\left(\theta,R\right),\overline{B}\left(\theta,r\right)\cap\overline{G}=\varnothing$ and

$$i\left(P\circ\left(\lambda N\right),B\left(\theta,r\right),K\right)+i\left(P\circ\left(\lambda N\right),G,K\right)\neq i\left(P\circ\left(\lambda N\right),B\left(\theta,R\right),K\right).$$

Therefore, for the operartor $P\circ\left(\lambda N\right)$, there is at least one fixed point in G and at least one in $B\left(\theta,R\right)\setminus\left[\overline{B}\left(\theta,r\right)\cup\overline{G}\right]$.

Finally, we prove that if $\left(u_0,v_0\right)$ is a non-negative solution of problem (1), then $u_0\not\equiv 0$ and $v_0\not\equiv 0$. Indeed, from condition (H5), assume $u_0\equiv 0$, then

$$\theta=-\Delta_p u_0+g_1\left(x,u_0\right)=\lambda f_1\left(x,u_0,v_0\right)\geq d_1 v_0^\eta\neq 0.$$

This is a contradiction. Thus, we have the proof.

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SỰ TỒN TẠI NGHIỆM YẾU KHÔNG ÂM KHÔNG TẦM THƯỜNG CỦA MỘT LỚP HỆ PHƯƠNG TRÌNH DẠNG LOGISTIC

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TÓM TẮT

Trong nghiên cứu này chúng tôi xét hệ phương trình có dạng logistic sau

$$\begin{cases} -\Delta_p u = \lambda f_1(x,u,v) - g_1(x,u) & \text{trong } \Omega, \\ -\Delta_p v = \lambda f_2(x,v,u) - g_2(x,v) & \text{trong } \Omega, \\ u = v = 0 & \text{trên } \partial\Omega, \end{cases}$$

Với giả thiết về thỏa mãn điều kiện về bậc tăng (của ẩn hàm) được chỉ ra sau của các hàm phi tuyến $f_i, g_i, i = 1, 2$ *. Chúng tôi chỉ ra sự tồn tại nghiệm yếu không âm cho hệ bằng phương pháp bậc tô pô kết hợp với lí luận về chặn dưới đơn điệu. Đây là một kết quả mở rộng cho các nghiên cứu trước đây.*

Từ khóa: bậc tô pô; hệ phương trình logistic; (p-1)-tuyến tính