TẠP CHÍ KHOA HỌC
TRƯỜNG ĐẠI HỌC SƯ PHẠM TP HỒ CHÍ MINH
Tập 23, Số 7 (2026): 1605-1615
Website: https://journal.hcmue.edu.vn

HO CHI MINH CITY UNIVERSITY OF EDUCATION
JOURNAL OF SCIENCE
Vol. 23, No. 7 (2026): 1605-1615
https://doi.org/10.54607/hcmue.js.23.7.4712(2026)

ISSN: 2734-9918

Research Article

CALDERÓN – ZYGMUND OPERATORS OF TYPE THETA ON A PAIR OF WEIGHTED LORENTZ SPACES

Tran Trung Toan, Phan Tuan Kiet, Nguyen Pham Nhat Minh, Tran Tri Dung [*]

Ho Chi Minh City University of Education, Vietnam

[*]*Corresponding author: Tran Tri Dung–Email: dungtt@hcmue.edu.vn*

Received: February 05, 2025; Revised: April 09, 2025; Accepted: April 15, 2025

ABSTRACT

In this paper, we study Calderón – Zygmund operators of type theta on weighted Lorentz spaces (see Definition 1.1, 1.3 and 1.4). More precisely, we present a sufficient condition on weight functions under which the boundedness of the Calderón – Zygmund operators of type theta on a pair of weighted Lorentz spaces is established (see Theorem 3.1). Our approach is to bound the norms of such operators using the boundedness of Hardy-type operators on weighted Lorentz spaces (see Lemma 2.2) and the boundedness of the Calderón – Zygmund operators of type theta on classical Lorentz spaces (see Proposition 2.3).

Keywords: Calderón – Zygmund operator of type theta; Hardy type operator; weight function; weighted Lorentz space

1. Introduction

The theory of Calderón – Zygmund operators plays a crucial role in modern harmonic analysis and PDEs. The boundedness of these operators on function spaces, which has been studied by many mathematicians, is one of the main topics in this theory. Coifman et al. (1974) established a well-known sufficient condition for the boundedness of Calderón – Zygmund operators on weighted Lebesgue spaces $L^p(u)$. Diening et al. (2003) showed that Calderón – Zygmund operators are bounded on generalized Lebesgue spaces $L^{p(\cdot)}(\mathbb{R}^n)$. Since then, these results have been extended to the setting of Lorentz spaces for not only Calderón – Zygmund operators but also various operators. For instance, the boundedness of the Calderón – Zygmund operators on weighted Lorentz spaces $\Lambda_u^p(\omega)$ was established by Carro et al. (2021). Recently, Minh et al. (2022) proved the boundedness of the Calderón – Zygmund operators of type θ on weighted Lorentz spaces $\Lambda_u^p(\omega)$. In addition, Rakotondratsimba (1998) established certain necessary and sufficient conditions for the

Cite this article as: Tran, T. T., Phan, T. K., Nguyen, P. N. M., & Tran, T. D. (2026). Calderón – Zygmund operators of type theta on a pair of weighted Lorentz spaces. *Ho Chi Minh City University of Education Journal of Science, 23*(7), 1605-1615. https://doi.org/10.54607/hcmue.js.23.7.4712(2026)

boundedness of fractional integral operators on weighted Lorentz spaces. The author showed that if fractional integral operators are bounded from weighted Lorentz spaces $L_v^{r,s}$ to weighted Lorentz spaces $L_u^{p,q}$ then $u(\cdot), v(\cdot)$ satisfy the following conditions

$$\left\| |\cdot|^{\alpha-n} \chi_{|\cdot|>R} \right\|_{L_u^{pq}} \left\| \frac{1}{v} \chi_{|\cdot|<R} \right\|_{L_v^{\frac{r}{r-1},\frac{s}{s-1}}} \le H$$

and

$$\left\| \frac{1}{v} |\cdot|^{\alpha-n} \chi_{|\cdot|>R} \right\|_{L_v^{\frac{r}{r-1},\frac{s}{s-1}}} \left\| \chi_{|\cdot|<R} \right\|_{L_u^{pq}} \le H^*$$

for all $R>0$, where χ_A denotes the characteristic function of the measurable set A. The author also demonstrated that the conditions above are not sufficient to imply the boundedness of fractional integral operators from $L_v^{r,s}$ into $L_u^{p,q}$. To obtain the converse statement, Rakotondratsimba (1998) further used the following pointwise inequality:

$$|x|^{n\left(\frac{\alpha}{n}+\frac{1}{p}-\frac{1}{r}\right)} \left(\sup_{4^{-1}|x|<|z|<4|x|} u(z) \right)^{\frac{1}{p}} \lesssim \left(v(x)\right)^{\frac{1}{r}} \quad for \ \ a.e. \ \ x$$

or $$|x|^{n\left(\frac{\alpha}{n}+\frac{1}{p}-\frac{1}{r}\right)} (u(x))^{\frac{1}{p}} \left(\sup_{4^{-1}|x|<|z|<4|x|} v(z)^{-\frac{1}{r}} \right) \le C \quad for \ a.e. \ x.$$

Motivated by the results mentioned above, in this paper, we study the boundedness of Calderón – Zygmund operators of type θ from weighted Lorentz spaces $L_v^{r,s}$ to weighted Lorentz spaces $L_u^{r,q}$. Specifically, we present sufficient conditions on the indices and weight functions of Lorentz spaces $L_v^{r,s}$ and $L_u^{r,q}$ to ensure the boundedness of Calderón – Zygmund operators of type θ. Throughout the paper, we assume that p,q,r,s satisfy the following conditions:

$$1 < r,s,p,q < \infty \text{ and } \max\{r,s\} \le \min\{p,q\}.$$

For convenience, we denote a constant by C. Moreover, if there exists a constant $C>0$ such that $A \le CB$ then we write $A \lesssim B$.

Definition 1.1. *(Weighted Lorentz spaces) Let* $1 \le p < \infty$, $1 \le q \le \infty$ *and* $u(\cdot)$ *be a weight function on* $\mathbb{R}^n$, *i.e., a nonnegative locally integrable function on* $\mathbb{R}^n$. *For a measurable function* f *on* $\mathbb{R}^n$, *we denote*

$$\|f\|_{L_u^{p,q}}=\begin{cases}\left(q\displaystyle\int_0^\infty\left(\int_{\{y\in\mathbb{R}^n:|f(y)|>t\}}u(y)dy\right)^{\frac{q}{p}}t^{q-1}dt\right)^{\frac{1}{q}}, \ if \ q<\infty\\ \sup\limits_{t>0}t\left(\displaystyle\int_{\{y\in\mathbb{R}^n:|f(y)|>t\}}u(y)dy\right)^{\frac{1}{p}}, \ if \ q=\infty.\end{cases}$$

Then the weighted Lorentz space $L_u^{p,q}$ *is defined by*

$$L_u^{p,q}=\left\{f:\|f\|_{L_u^{p,q}}<\infty\right\}.$$

Definition 1.2. *(l^p space) Let* $p\in[1;\infty)$. *Then* l^p *is the space consisting of all sequences of real numbers* $x=(x_k)_k$ *satisfying*

$$\sum_k|x_k|^p<\infty.$$

We define a norm on l^p *as the real-valued function* $\|\cdot\|_p$ *defined by*

$$\|x\|_p=\left(\sum_k|x_k|^p\right)^{\frac{1}{p}}, \text{ for every } x=(x_k)_k\in l^p.$$

Definition 1.3. *(Yabuta, 1985) Let* θ *be a nonnegative, nondecreasing function on* $(0;\infty)$ *with* $\int_0^1\theta(t)t^{-1}dt<\infty$. *Suppose* $K(x,y)$ *is a continuous function defined on* $\mathbb{R}^n\times\mathbb{R}^n\setminus\{(x,x):x\in\mathbb{R}^n\}$ *and satisfies the following conditions:*

(i) $|K(x,y)|\le\dfrac{C}{|x-y|^n}$ *for* $x\neq y$;

(ii) $|K(x,y)-K(z,y)|+|K(y,x)-K(y,z)|\le\dfrac{C}{|x-y|^n}\theta\left(\dfrac{|x-z|}{|x-y|}\right)$

for every x,y,z *with* $2|x-z|\le|x-y|$.

Then $K(x,y)$ *is called a standard kernel of type* θ.

Definition 1.4. *(Yabuta, 1985) Let* θ *be a function as in Definition 1.3. Suppose* T *is a linear operator from* $\mathcal{S}(\mathbb{R}^n)$ *to* $\mathcal{S}'(\mathbb{R}^n)$, *satisfying the following conditions:*

(i) T *is bounded on* $L^2(\mathbb{R}^n)$, *i.e.,*

$$\|Tf\|_{L^2}\le C\|f\|_{L^2} \text{ for every } f\in C_c^\infty(\mathbb{R}^n);$$

(ii) T *has the representation*

$$Tf(x)=\int_{\mathbb{R}}K(x,y)f(y)dy, \text{ for every } f\in C_c^\infty(\mathbb{R}^n) \text{ and } x\notin supp(f),$$

where K(x,y) is a standard kernel of type θ.

Then T *is called a Calderón-Zygmund operator of type* θ.

2. Key Lemmas

Lemma 2.1. Let $\{E_k\}$ be a sequence of measurable sets satisfying $\sum_k \chi_{E_k} \lesssim \chi_{\cup E_k}$. Then

1) $\sum_k \left\| f\chi_{E_k} \right\|_{L_v^{r,s}}^{\lambda} \lesssim \left\| f\chi_{\cup E_k} \right\|_{L_v^{r,s}}^{\lambda}$ for all functions f,

where λ is a real number such that $max(r,s) \le \lambda$.

2) $\left\| \sum_k f\chi_{E_k} \right\|_{L_u^{p,q}}^{\gamma} \lesssim \sum_k \left\| f\chi_{E_k} \right\|_{L_u^{p,q}}^{\gamma}$ for all functions f,

where γ is a real number such that $0 < \gamma \le \min(p,q)$.

Proof.

1) For an arbitrary k, we have

$$\left\| f\chi_{E_k} \right\|_{L_v^{r,s}}^{\lambda} = \left[s\int_0^{\infty} \left(\int_{\{y:|f(y).\chi_{E_k}(y)|>t\}} v(y)dy \right)^{\frac{s}{r}} t^{s-1}dt \right]^{\frac{\lambda}{s}} = \left[s\int_0^{\infty} \left(\int_{\{y:|f(y)|>t\}\cap E_k} v(y)dy \right)^{\frac{s}{r}} t^{s-1}dt \right]^{\frac{\lambda}{s}}.$$

As $\max(r,s) \le \lambda$, by Minkowski's inequality, we obtain

$$\sum_k \left\| f\chi_{E_k} \right\|_{L_v^{r,s}}^{\lambda} = \sum_k \left[s\int_0^{\infty} \left(\int_{\{y:|f(y)|>t\}\cap E_k} v(y)dy \right)^{\frac{s}{r}} t^{s-1}dt \right]^{\frac{\lambda}{s}} \le \left[s\int_0^{\infty} \left(\sum_k \left(\int_{\{y:|f(y)|>t\}\cap E_k} v(y)dy \right)^{\frac{\lambda}{r}} \right)^{\frac{s}{\lambda}} t^{s-1}dt \right]^{\frac{\lambda}{s}}$$

$$\lesssim \left[s\int_0^{\infty} \left(\int_{\{y:|f(y)|>t\}} v(y)dy \right)^{\frac{s}{r}} t^{s-1}dt \right]^{\frac{\lambda}{s}} = \left\| f \right\|_{L_v^{r,s}}^{\lambda}.$$

2) For $p = q$ or $q < p$, $\left\| \cdot \right\|_{L_u^{p/\gamma,q/\gamma}}$ is equivalent to a norm and hence

$$\left\| f\sum_k \chi_{E_k} \right\|_{L_u^{p,q}}^{\gamma} = \left\| \left(f\sum_k \chi_{E_k} \right)^{\gamma} \right\|_{L_u^{p/\gamma,q/\gamma}} \lesssim \left\| f^{\gamma}.\sum_k \chi_{E_k} \right\|_{L_u^{p/\gamma,q/\gamma}} \lesssim \sum_k \left\| f^{\gamma}\chi_{E_k} \right\|_{L_u^{p/\gamma,q/\gamma}} \lesssim \sum_k \left\| f.\chi_{E_k} \right\|_{L_u^{p,q}}^{\gamma}.$$

For $p < q$, we have $\gamma \le p < q$ or $\frac{\gamma}{p} \le 1$, so

$$\left\| f\sum_k \chi_{E_k} \right\|_{L_u^{p,q}}^{\gamma} = \left[q\int_0^{\infty} \left(\int_{\left\{y:\left|f(y).\sum_k \chi_{E_k}(y)\right|>t\right\}} u(y)d(y) \right)^{q/p} t^{q-1}dt \right]^{\gamma/q} \lesssim \left[q\int_0^{\infty} \left(\sum_k \int_{E_k\cap\{|f(y)|>t\}} u(y)dy \right)^{q/p} t^{q-1}dt \right]^{\gamma/q}$$

$$\lesssim\left[\sum_j q\int_{2^j}^{2^{j+1}}\left(\sum_k \int_{E_k\cap\{y:|f(y)|>2^j\}} u(y)dy\right)^{q/p} t^{q-1}dt\right]^{\gamma/q} \lesssim\left[\sum_j 2^{jq}\left(\sum_k \int_{E_k\cap\{y:|f(y)|>2^j\}} u(y)dy\right)^{q/p}\right]^{\gamma/q}$$

$$\lesssim\left[\sum_j\left(2^{j\gamma}\left(\sum_k \int_{E_k\cap\left\{y:|f(y)|>2^j\right\}} u(y)dy\right)^{\gamma/p}\right)^{q/\gamma}\right]^{\gamma/q} \lesssim\left[\sum_j\left(2^{j\gamma}\sum_k\left(\int_{E_k\cap\left\{y:|f(y)|>2^j\right\}} u(y)dy\right)^{\gamma/p}\right)^{q/\gamma}\right]^{\gamma/q}.$$

Take any $\varepsilon>0$, there exists a nonnegative sequence $(b_j)_j\in l^{\frac{q}{q-\gamma}}$ with $\sum_j b_j^{\frac{q}{q-\gamma}}\le 1$ such that

$$\left[\sum_j\left(2^{j\gamma}\sum_k\left(\int_{E_k\cap\left\{y:|f(y)|>2^j\right\}} u(y)dy\right)^{\gamma/p}\right)^{q/\gamma}\right]^{\gamma/q}-\varepsilon \le\sum_k\sum_j 2^{j\gamma}b_j\left(\int_{E_k\cap\left\{y:|f(y)|>2^j\right\}} u(y)dy\right)^{\gamma/p}$$

$$\lesssim\sum_k\left(\sum_j 2^{jq}\left(\int_{E_k\cap\left\{y:|f(y)|>2^j\right\}} u(y)dy\right)^{q/p}\right)^{\gamma/q},$$

where the last estimate is derived from Hölder's inequality with exponents $\frac{q}{\gamma}$ and $\frac{q}{q-\gamma}$.

By letting $\varepsilon\to 0^+$, we deduce that

$$\left[\sum_j\left(2^{j\gamma}\sum_k\left(\int_{E_k\cap\left\{y:|f(y)|>2^j\right\}} u(y)dy\right)^{\gamma/p}\right)^{q/\gamma}\right]^{\gamma/q}\lesssim\sum_k\left(\sum_j 2^{jq}\left(\int_{E_k\cap\left\{y:|f(y)|>2^j\right\}} u(y)dy\right)^{q/p}\right)^{\gamma/q}.$$

Therefore,

$$\left\|f\sum_k\chi_{E_k}\right\|_{L_u^{p,q}}^{\gamma}\lesssim\sum_k\left(\sum_j 2^{jq}\left(\int_{E_k\cap\left\{y:|f(y)|>2^j\right\}} u(y)dy\right)^{q/p}\right)^{\gamma/q}\lesssim\sum_k\left\|f\cdot\chi_{E_k}\right\|_{L_u^{p,q}}^{\gamma}. \qquad \square$$

For measurable functions $a(\cdot), b(\cdot)$ on $\mathbb{R}^n$, the Hardy-type operators $H_{a,b}, H_{a,b}^*$ are defined by

$$H_{a,b}f(x)=a(x)\int_{\{y:|y|\le|x|\}} f(y)b(y)dy$$

and

$$H_{a,b}^*g(x)=b(x)\int_{\{y:|y|\ge|x|\}} g(y)a(y)dy .$$

We recall here a result concerning the boundedness of Hardy-type operators on weighted Lorentz spaces, which plays an important role in the proof of our main result.

Lemma 2.2. *(Rakotondratsimba, 1998, Lemma 3)* Let $H_{a,b}, H_{a,b}^*$ be Hardy-type operators.

Then

1) $\left\| H_{a,b} f \right\|_{L_u^{p,q}} \lesssim \| f \|_{L_v^{r,s}}$, $\forall f \ge 0$

is equivalent to

$$\sup_{R>0} \left\| a.\chi_{|\cdot|>R} \right\|_{L_u^{p,q}} \left\| \frac{b}{v} .\chi_{|\cdot|<R} \right\|_{L_v^{\frac{r}{r-1},\frac{s}{s-1}}} < \infty.$$

2) $\left\| H_{a,b}^{*} g \right\|_{L_u^{p,q}} \lesssim \| g \|_{L_v^{r,s}}$, $\forall g \ge 0$

is equivalent to

$$\sup_{R>0} \left\| b.\chi_{|\cdot|<R} \right\|_{L_u^{p,q}} \left\| \frac{a}{v} .\chi_{|\cdot|>R} \right\|_{L_v^{\frac{r}{r-1},\frac{s}{s-1}}} < \infty.$$

Proof.

1) The proof of this lemma is provided in Edmunds et al. (1994). However, for convenience of the reader, we recall the proof here.

Suppose that $\left\| H_{a,b} f \right\|_{L_u^{p,q}} \lesssim \| f \|_{L_v^{r,s}}$ for all $f \ge 0$.

For any $\| f \|_{L_v^{r,s}} \le 1$ and $R > 0$, we have

$$\left(\int_{\{y:|y|\le R\}} f(y)b(y)dy \right) . \left\| a.\chi_{|\cdot|>R} \right\|_{L_u^{p,q}} \le \left\| a(\cdot) \int_{\{y:|y|\le|\cdot|\}} f(y)b(y)dy.\chi_{|\cdot|>R} \right\|_{L_u^{p,q}}$$

$$= \left\| H_{a,b} f .\chi_{|\cdot|>R} \right\|_{L_u^{p,q}} \lesssim \| f \|_{L_v^{r,s}} \le 1.$$

Taking the supremum over all values $R > 0$ and f with $\| f \|_{L_v^{r,s}} \le 1$, we obtain

$$\sup_{R>0} \left\| \frac{b}{v} .\chi_{|\cdot|<R} \right\|_{L_v^{\frac{r}{r-1},\frac{s}{s-1}}} . \left\| a.\chi_{|\cdot|>R} \right\|_{L_u^{p,q}} \le C < +\infty.$$

Conversely, assume that $\sup_{R>0} \left\| a.\chi_{|\cdot|>R} \right\|_{L_u^{p,q}} \left\| \frac{b}{v} .\chi_{|\cdot|<R} \right\|_{L_v^{\frac{r}{r-1},\frac{s}{s-1}}} < \infty.$

Let $f \ge 0$ be arbitrary. Choose $m \in \mathbb{Z}$ such that $\int_{\mathbb{R}^n} f(y)b(y)dy \in \left(2^m ; 2^{m+1}\right]$ (in case $\int_{\mathbb{R}^n} f(y)b(y)dy = \infty$, we choose $m = +\infty$).

Then, there exists an increasing sequence $\{x_k\}_{k=-\infty}^{m}$ such that

$$2^k = \int_{\{y:|y|\le x_k\}} f(y)b(y)dy = \int_{\{y:x_k\le|y|<x_{k+1}\}} f(y)b(y)dy,$$

and $2^m = \int_{\{y:|y|\le x_m\}} f(y)b(y)dy$. Set $x_{m+1} = \infty$ and $E_k = \{ y : x_k \le |y| < x_{k+1} \}$. Then $\{E_k\}_k$ is a family of disjoint sets and $\bigcup_{k\le m} E_k = \mathbb{R}^n$.

It follows that $H_{a,b}(x) = a(x)\int_{\{y:|y|\le|x|\}} f(y)b(y)dy \le a(x).2^{k+1}, \forall x \in E_k$.

For $\max\{r,s\} \le \lambda \le \min\{p,q\}$, by Lemma 2.1, we have

$$\|H_{a,b}f\|^{\lambda}_{L^{p,q}_u} = \left\|\sum_{k\le m} H_{a,b}f.\chi_{E_k}\right\|^{\lambda}_{L^{p,q}_u} \lesssim \sum_{k\le m}\left\|H_{a,b}f.\chi_{E_k}\right\|^{\lambda}_{L^{p,q}_u} \le \sum_{k\le m} 2^{(k+1)\lambda}\left\|a.\chi_{E_k}\right\|^{\lambda}_{L^{p,q}_u} \le 4^{\lambda}.\sum_{k\le m} 2^{(k-1)\lambda}\left\|a.\chi_{E_k}\right\|^{\lambda}_{L^{p,q}_u}$$

$$\le 4^{\lambda}.\sum_{k\le m}\left(\int_{\{y:x_{k-1}\le|y|<x_k\}} f(y)b(y)dy\right)^{\lambda}.\left\|a.\chi_{|\cdot|>x_k}\right\|^{\lambda}_{L^{p,q}_u} \lesssim 4^{\lambda}.\sum_{k\le m}\left\|f.\chi_{E_{k-1}}\right\|^{\lambda}_{L^{r,s}_v}.\left\|\frac{b}{v}.\chi_{|\cdot|<x_k}\right\|^{\lambda}_{L_v^{\frac{r}{r-1},\frac{s}{s-1}}}.\left\|a.\chi_{|\cdot|>x_k}\right\|^{\lambda}_{L^{p,q}_u}$$

$$\lesssim 4^{\lambda}.\sum_{k\le m}\left\|f.\chi_{E_k}\right\|^{\lambda}_{L^{r,s}_v} \lesssim \left(4\|f\|_{L^{r,s}_v}\right)^{\lambda}.$$

Therefore $\|H_{a,b}f\|_{L^{p,q}_u} \lesssim \|f\|_{L^{r,s}_v}$.

2) Notice that $H_{a,b}$ and $H^*_{a,b}$ are associated operators. Thus, the boundedness $H^*_{a,b}: L^{r,s}_v \to L^{p,q}_u$ is equivalent to $H_{a,b}: L_u^{\frac{p}{p-1},\frac{q}{q-1}}\left(\frac{1}{u}\right) \to L_v^{\frac{r}{r-1},\frac{s}{s-1}}\left(\frac{1}{v}\right)$, which denotes the boundedness $\left\|\frac{1}{v}.H_{a,b}f\right\|_{L_v^{\frac{r}{r-1},\frac{s}{s-1}}} \lesssim \left\|\frac{1}{u}f\right\|_{L_u^{\frac{p}{p-1},\frac{q}{q-1}}}$. Therefore, part (2) follows analogously from part (1). □

Proposition 2.3. *(Thai et al., 2022, Theorem 2.4)* Assume that T is a Calderón – Zygmund operator of type θ. Then T is bounded on $L^{r,s}_1$.

Lemma 2.4. *(Rakotondratsimba, 1998, Lemma 5)* Let T be a Calderón – Zygmund operator of type θ and $0 < \lambda \le \min\{p,q\}$. Then

$$\|Tf\|^{\lambda}_{L^{p,q}_u} \lesssim S_1^{\lambda} + S_2^{\lambda} + S_3^{\lambda}, \forall f \in C_c^{\infty}(\mathbb{R}^n)$$

where

$$S_1^{\lambda} = \left\||\cdot|^{-n}\int_{|y|\le|\cdot|}|f(y)|dy\right\|^{\lambda}_{L^{p,q}_u},$$

$$S_2^{\lambda} = \sum_{k\in\mathbb{Z}}\left\|T(f\chi_{G_k}).\chi_{E_k}\right\|^{\lambda}_{L^{p,q}_u},$$

$$S_3^{\lambda} = \left\|\int_{|\cdot|\le|y|}|f(y)|.|y|^{-n}dy\right\|^{\lambda}_{L^{p,q}_u}$$

and $E_k = \{x: 2^k < |x| \le 2^{k+1}\}, G_k = \{x: 2^{k-1} < |x| \le 2^{k+2}\}$.

3. Main result

Theorem 3.1. *Let T be a Calderón – Zygmund operator of type θ. Suppose that $s \le r \le q$ and $u(\cdot), v(\cdot)$ satisfy the following conditions*

$$\left\||\cdot|^{-n}.\chi_{\{|\cdot|>R\}}\right\|_{L^{r,q}_u}.\left\|\frac{1}{v}.\chi_{\{|\cdot|<R\}}\right\|_{L_v^{\frac{r}{r-1},\frac{s}{s-1}}} \le H,\ \forall R > 0; \tag{1.1}$$

$$\left\| \frac{1}{v} |\cdot|^{-n} \chi_{\{|\cdot|>R\}} \right\|_{L_v^{\frac{r}{r-1},\frac{s}{s-1}}} \cdot \left\| \chi_{\{|\cdot|<R\}} \right\|_{L_u^{r,q}} \le H^*, \ \forall R > 0; \tag{1.2}$$

$$\sup_{4^{-1}|x|<|z|<4|x|} u(z) < v(x), \ a.e \ x \in \mathbb{R}^n. \tag{1.3}$$

Then T is bounded from $L_v^{r,s}$ *to* $L_u^{r,q}$.

Proof. In view of Lemma 2.4, we have

$$\|Tf\|_{L^{r,q}}^r \lesssim S_1^r + S_2^r + S_3^r,$$

where

$$S_1^r = \left\| |\cdot|^{-n} \int_{|y|\le|\cdot|} |f(y)|\, dy \right\|_{L_u^{r,q}}^r,$$

$$S_2^r = \sum_{k\in\mathbb{Z}} \left\| T(f\chi_{G_k}).\chi_{E_k} \right\|_{L_u^{r,q}}^r,$$

$$S_3^r = \left\| \int_{|\cdot|\le|y|} |f(y)|.|y|^{-n}\, dy \right\|_{L_u^{r,q}}^r$$

and $E_k = \{x : 2^k < |x| \le 2^{k+1}\}, G_k = \{x : 2^{k-1} < |x| \le 2^{k+2}\}$.

We first estimate S_1^r. Choose $a(\cdot) = |\cdot|^{-n}, b(\cdot) = 1$. Then by (1.1), we have

$$\sup_{R>0} \left\| a.\chi_{\{|\cdot|>R\}} \right\|_{L_u^{r,q}} \cdot \left\| \frac{b}{v}.\chi_{\{|\cdot|<R\}} \right\|_{L_v^{\frac{r}{r-1},\frac{s}{s-1}}} < \infty.$$

So, by Lemma 2.2, we obtain $\left\| H_{a,b} f \right\|_{L_u^{p,q}} \lesssim \|f\|_{L_v^{r,s}}$, or equivalently

$$\left\| |\cdot|^{-n} \int_{|y|\le|\cdot|} |f(y)|\, dy \right\|_{L_u^{r,q}} \lesssim \|f\|_{L_v^{r,s}}.$$

Thus $S_1^r \lesssim \|f\|_{L_v^{r,s}}^r$.

S_3^r can be estimated in a similar way. Indeed, it follows from (1.2) that

$$\sup_{R>0} \left\| \frac{a}{v} \chi_{\{|\cdot|>R\}} \right\|_{L_v^{\frac{r}{r-1},\frac{s}{s-1}}} \cdot \left\| b.\chi_{\{|\cdot|<R\}} \right\|_{L_u^{r,q}} < \infty,$$

which implies $\left\| H_{a,b}^* f \right\|_{L_u^{r,q}} \lesssim \|f\|_{L_v^{r,s}}$, or equivalently $\left\| \int_{|\cdot|\le|y|} |f(y)|.|y|^{-n}\, dy \right\|_{L_u^{r,q}} \lesssim \|f\|_{L_v^{r,s}}$.

Thus $S_3^r \lesssim \|f\|_{L_v^{r,s}}^r$.

Finally, we estimate S_2^r. For any $k \in \mathbb{Z}$, we denote $\mathcal{U}_k = \sup_{z\in E_k} u(z)$. Then

$$\left\|T\left(f.\chi_{G_k}\right).\chi_{E_k}\right\|_{L_u^{r,q}}=\left(q\int_0^\infty\left(\int_{\left\{y:\left|T\left(f.\chi_{G_k}\right).\chi_{E_k}\right|>t\right\}}u(y)dy\right)^{\frac{q}{r}}t^{q-1}dt\right)^{\frac{1}{q}}$$

$$\lesssim \mathcal{U}_k^{\frac{1}{r}}\left(q\int_0^\infty\left(\int_{\left\{y:\left|T\left(f.\chi_{G_k}\right).\chi_{E_k}\right|>t\right\}}dy\right)^{\frac{q}{r}}t^{q-1}dt\right)^{\frac{1}{q}}$$

$$\lesssim \mathcal{U}_k^{\frac{1}{r}}.\left\|T\left(f.\chi_{G_k}\right).\chi_{E_k}\right\|_{L_1^{r,q}}\lesssim \mathcal{U}_k^{\frac{1}{r}}.\left\|T\left(f.\chi_{G_k}\right)\right\|_{L_1^{r,s}}.$$

At this stage, it follows from Proposition 2.3 and (1.3) that

$$\left\|T(f.\chi_{G_k}).\chi_{E_k}\right\|_{L_u^{r,q}}\lesssim \mathcal{U}_k^{\frac{1}{r}}.\left\|T(f\chi_{G_k})\right\|_{L_1^{r,s}}\lesssim \mathcal{U}_k^{\frac{1}{r}}.\left\|f\chi_{G_k}\right\|_{L_1^{r,s}}$$

$$\lesssim \mathcal{U}_k^{\frac{1}{r}}.\left(s\int_0^\infty\left(\int_{G_k\cap\{y:|f(y)|>t\}}dy\right)^{\frac{s}{r}}t^{s-1}dt\right)^{\frac{1}{s}}\lesssim\left(s\int_0^\infty\left(\int_{G_k\cap\{y:|f(y)|>t\}}\left(\sup_{z\in E_k}u(z)\right)dy\right)^{\frac{s}{r}}t^{s-1}dt\right)^{\frac{1}{s}}$$

$$\lesssim\left(s\int_0^\infty\left(\int_{G_k\cap\{y:|f(y)|>t\}}\left(\sup_{4^{-1}|y|<|z|<4|y|}u(z)\right)dy\right)^{\frac{s}{r}}t^{s-1}dt\right)^{\frac{1}{s}}$$

$$\lesssim\left(s\int_0^\infty\left(\int_{G_k\cap\{y:|f(y)|>t\}}v(y)dy\right)^{\frac{s}{r}}t^{s-1}dt\right)^{\frac{1}{s}}\lesssim\left\|f.\chi_{G_k}\right\|_{L_v^{r,s}}.$$

Finally, by using Lemma 2.1, we deduce that

$$S_2^\lambda\lesssim\sum_k\left\|f\chi_{G_k}\right\|_{L_v^{r,s}}^\lambda\lesssim\|f\|_{L_v^{r,s}}^\lambda.$$

Therefore,

$$\|Tf\|_{L_u^{r,q}}\lesssim\|f\|_{L_v^{r,s}},$$

that is, T is bounded from $L_v^{r,s}$ to $L_u^{r,q}$. □

4. Conclusion

In summary, we have established certain sufficient conditions on the indices and weight functions under which the boundedness of Calderón – Zygmund operators of type theta on weighted Lorentz spaces holds. More specifically, our main result is the following theorem:

Theorem 3.1. *Let* T *be a Calderón – Zygmund operator of type* θ. *Suppose that* $s \le r \le q$ *and* $u(\cdot), v(\cdot)$ *satisfy the following conditions*

$$\left\| |\cdot|^{-n} \cdot \chi_{\{|\cdot|>R\}} \right\|_{L_u^{r,q}} \cdot \left\| \frac{1}{v} \cdot \chi_{\{|\cdot|<R\}} \right\|_{L_v^{\frac{r}{r-1},\frac{s}{s-1}}} \le H, \ \forall R > 0; \tag{1.1}$$

$$\left\| \frac{1}{v} |\cdot|^{-n} \chi_{\{|\cdot|>R\}} \right\|_{L_v^{\frac{r}{r-1},\frac{s}{s-1}}} \cdot \left\| \chi_{\{|\cdot|<R\}} \right\|_{L_u^{r,q}} \le H^*, \ \forall R > 0; \tag{1.2}$$

$$\sup_{4^{-1}|x|<|z|<4|x|} u(z) < v(x), \ a.e \ x \in \mathbb{R}^n. \tag{1.3}$$

Then T is bounded from $L_v^{r,s}$ *to* $L_u^{r,q}$.

- ***Conflict of Interest:*** *Authors have no conflict of interest to declare.*
- ***Acknowledgments:*** *This research is funded by Ho Chi Minh City University of Education, Foundation for Science and Technology under grant number CS.2024.19.24TĐ.*

REFERENCES

Carro, M., & Soria, J. (1997). The Hardy–Littlewood maximal function and weighted Lorentz spaces. *Journal of the London Mathematical Society*, *55*(1), 146–158. https://doi.org/10.1112/S0024610796004462

Chanillo, S. (1982). A note on commutators. *Indiana University Mathematics Journal*, *31*(1), 7–16. https://doi.org/10.1512/iumj.1982.31.31002

Coifman, R. R., & Fefferman, C. (1974). Weighted norm inequalities for maximal functions and singular integrals. *Stud. Math.*, *51*, 241–250.

Cruz-Uribe, D., Fiorenza, A., Martell, J. M., & Pérez, C. (2006). The boundedness of classical operators on variable L P spaces. *Annales Academiae Scientiarum Fennicae. Mathematica*, *31*, 139–264.

Diening, L., & Růžička, M. (2003). Calderón–Zygmund operators on generalized Lebesgue spaces and problems related to fluid dynamics. *Journal für die Reine und Angewandte Mathematik*, *563*, 197–220. https://doi.org/10.1515/crll.2003.081

Edmunds, D. E., Gurka, P., & Pick, L. (1994). Compactness of Hardy-type integral operators in weighted Banach function spaces. *Studia Mathematica*, *109*, 73–90.

Grafakos, L. (2009). *Modern Fourier analysis* (Vol. 250, pp. xvi+-504). Springer.

Hunt, A. (1966). On L(p, q) spaces. *L'enseign. Math.*, *12*, 249–294. https://doi.org/10.5169/seals-40747

Kokilashvili, V., & Krbec, M. (1991). *Weighted inequalities in Lorentz and Orlicz spaces* (2nd ed.). World Scientific, 57.

Rakotondratsimba, Y. (1998). On the boundedness of classical operators on weighted Lorentz spaces. *Georgian Mathematical Journal*, *5*(2), 177–200. https://doi.org/10.1515/GMJ.1998.177

Sawyer, E. (1984). Weighted Lebesgue and Lorentz norm inequality for the Hardy operators. *Trans. Amer. Math. Soc.*, *281*, 329–337. https://doi.org/10.1090/s0002-9947-1984-0719673-4

Thai, H. M., Nguyen, V. T. D., Hoang, N. P., & Tran, T. D. (2022). The boundedness of Calderón-Zygmund operators of type theta on generalized weighted Lorentz spaces. *Ho Chi Minh City University of Education Journal of Science*, *19*(6), 844–855. https://doi.org/10.54607/hcmue.js.19.7.3362(2022)

Yabuta, K. (1985). Calderón-Zygmund operators and pseudo-differential operators. *Communications in Partial Differential Equations*, *10*(9), 1005–1022. https://doi.org/10.1080/03605308508820398

TOÁN TỬ CALDERÓN – ZYGMUND LOẠI THETA TRÊN MỘT CẶP KHÔNG GIAN HÀM LORENTZ CÓ TRỌNG

Trần Trung Toàn, Phan Tuấn Kiệt, Nguyễn Phạm Nhật Minh, Trần Trí Dũng*

Trường Đại học Sư phạm Thành phố Hồ Chí Minh, Việt Nam

**Tác giả liên hệ: Trần Trí Dũng – Email: dungtt@hcmue.edu.vn*

Ngày nhận bài: 05-02-2025; Ngày nhận bài sửa: 09-4-2025; Ngày duyệt đăng: 15-4-2025

TÓM TẮT

Trong bài báo này, chúng tôi nghiên cứu các toán tử Calderón – Zygmund loại theta trên các không gian Lorentz có trọng (xem các Definition 1.1, 1.3, 1.4). Chính xác hơn, chúng tôi đưa ra một điều kiện đủ đối với các hàm trọng để thu được tính bị chặn của các toán tử Calderón – Zygmund loại theta trên một cặp không gian Lorentz có trọng (xem Theorem 3.1). Cách tiếp cận của chúng tôi để chặn chuẩn của các toán tử như vậy là sử dụng tính bị chặn của các toán tử loại Hardy trên không gian Lorentz có trọng (xem Bổ đề 2.2) và tính bị chặn của các toán tử Calderón–Zygmund loại theta trên không gian Lorentz cổ điển (xem Mệnh đề 2.3).

Từ khóa: toán tử Calderón – Zygmund loại theta; toán tử loại Hardy; hàm trọng; không gian Lorentz có trọng