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Research Article

GRADIENT-BASED METHODS FOR NOISY QUADRATIC OPTIMIZATION

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ABSTRACT

In this paper, we study the problem of minimizing a noisy multivariate quadratic function using an inexact gradient descent framework, where the gradients are approximated via central finite difference schemes. Two noise models are analyzed: (i) the quadratic matrix is known, with noise in the linear and constant terms; (ii) both the matrix and linear term are known, with noise only in the constant term. We derive explicit error bounds, establish convergence rates using Polyak's estimates (Polyak, 1987), and determine the iteration complexity of the sequence generated by the inexact gradient descent method with a fixed step size.

Keywords: central finite difference scheme; derivative-free optimization; inexact gradient descent method; noisy multivariate quadratic functions

1. Introduction

In this paper, we study the problem of minimizing a noisy multivariate quadratic function given in the form

$$f_q(x) = \frac{1}{2}\langle Ax, x\rangle + \langle b, x\rangle + c, x \in \mathbb{R}^n \qquad (1.1)$$

where $A \in \mathbb{R}^{n\times n}$ is a symmetric positive definite matrix, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$. In practical settings, the exact form of f_q is not accessible, and only a noisy approximation ϕ_q of f_q is known. We analyze two noise models for ϕ_q as follows:

i. Type I: The observed function is modeled as

$$\phi_q(x) = \frac{1}{2}\langle Ax, x\rangle + \langle \tilde{b}(x), x\rangle + \tilde{c}(x), \qquad (1.2)$$

where $\tilde{b}(x) = b + \xi_b(x), \tilde{c}(x) = c + \xi_c(x)$ and $\xi_b : \mathbb{R}^n \to \mathbb{R}^n, \xi_c : \mathbb{R}^n \to \mathbb{R}.$

In this case, the following conditions hold:

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- The matrix A, vector $\tilde{b}(x)$, and $\tilde{c}(x)$ are all known.
- The perturbations $\tilde{b}(x)$ and $\tilde{c}(x)$ are assumed to be bounded, with constants $\overline{\xi_b} > 0$

and $\overline{\xi_c} > 0$ such that:

$$\|\xi_b(x)\| \leq \overline{\xi_b}, \quad |\xi_c(x)| \leq \overline{\xi_c}, \quad \text{for all } x \in \mathbb{R}^n.$$

- It is assumed that an upper bound $\theta > 0$ for $\|b\|$ is known, i.e.,

$$\|b\| \leq \theta.$$

ii. Type II: The observed function is modeled as

$$\phi_q(x) = \frac{1}{2}\langle Ax, x\rangle + \langle b, x\rangle + \tilde{c}(x), \qquad (1.3)$$

where $\tilde{c}(x) = c + \xi_c(x)$ with $\xi_c : \mathbb{R}^n \to \mathbb{R}$.

In this case, the following conditions hold:

- The matrix A, vector b, and $\tilde{c}(x)$ are all known.
- The perturbation $\xi_c(x)$ is assumed to be bounded, with a constant $\overline{\xi_c} > 0$ such that:

$$|\xi_c(x)| \leq \overline{\xi_c}, \quad \text{for all } x \in \mathbb{R}^n.$$

We observe that the problem class described in (1.1) concerns the minimization of a strongly convex and continuously differentiable function. To address this minimization problem under the noisy conditions given in Type I and Type II, we consider an *inexact gradient method (the gradient method in the presence of noise)* described in (Polyak, 1987): Given an initial point $x^0 \in \mathbb{R}^n$ and a fixed step size $t > 0$, we construct the following iterative scheme:

$$x^{k+1} := x^k - t \cdot g^k, \text{with } g^k = \nabla f_q(x^k) + r^k \quad (1.4)$$

where g^k is an *approximation* of the gradient $\nabla f_q(x^k)$, and the error term $r^k \in \mathbb{R}^n$ satisfies

$$\|r^k\| = \|g^k - \nabla f_q(x^k)\| \leq \gamma \quad \text{for all } k \in \mathbb{N},$$

with a given constant $\gamma > 0$. If the step size t satisfies $0 < t < \frac{2}{\|A\|}$, the inexact gradient iteration k satisfies the inequality

$$\|x^k - x^*\| \leq \frac{t\gamma}{1-\rho} + \rho^k\left(\|x^0 - x^*\| - \frac{t\gamma}{1-\rho}\right), (1.5)$$

where $\rho = \|I - tA\|$ and x^* is the unique minimizer of f_q (Polyak, 1987).

The main objective of this paper is to apply the inexact gradient descent method with fixed step size as described in (1.4) to solve (1.1) and to derive the choice of step size along with explicit expressions for γ in inequality (1.5), based on the known noise bounds $\overline{\xi_b}$ and

$\bar{\xi}_c$ for Type I and Type II. Moreover, using inequality (1.5), we establish an upper bound on the optimality gap $f(x^k) - f(x^*)$ after k iterations.

The remainder of the paper is organized as follows. In Section 2, we present the necessary preliminaries and notations, while in Section 3, we present the inexact gradient descent method and provide a detailed convergence analysis under both Type I and Type II noise models.

2. Preliminaries and Notations

We begin by recalling some concepts and notation used throughout this paper. All our considerations are given in the space $\mathbb{R}^n$ with the Euclidean norm $\|\cdot\|$ given by

$$\|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2},$$

where $x = (x_1, x_2, \ldots, x_n)$ is a vector in $\mathbb{R}^n$.

For each $i = 1, \ldots, n$, let e_i denote the i^{th} basis vector in $\mathbb{R}^n$. We use $\mathbb{N}$ to denote the set of nonnegative integers. For any $m \times n$ matrix A, the matrix norm is defined by

$$\|A\| := \max\left\{\|Ax\| \mid \|x\| = 1,\, x \in \mathbb{R}^n\right\}.$$

Definition 2.1. *The mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ is called Lipschitz continuous if there is some constant $L \geq 0$ such that*

$$\|f(x) - f(y)\| \leq L\|x - y\|, \quad \text{for all } x, y \in \mathbb{R}^n.$$

Proposition 2.2. *Let f_q be the quadratic function defined in* (1.1) *. Then, for all $x \in \mathbb{R}^n$, $\nabla f_q(x) = Ax + b$ and $\nabla^2 f_q(x) = A$. Consequently, the gradient mapping ∇f_q is Lipschitz continuous with the Lipschitz constant $L = \|A\|$ and the Hessian operator $\nabla^2 f_q$ is Lipschitz continuous with the Lipschitz constant $M = 0$.*

The following result, which is called the descent lemma, is taken from Izmailov and Solodov(2014).

Lemma 2.3. *For any $f : \mathbb{R}^n \to \mathbb{R}$ and any $x, y \in \mathbb{R}^n$, if f is differentiable on the line segment $[x, y]$ with its derivative being Lipschitz continuous on this segment with a constant $L > 0$, then*

$$| f(y) - f(x) - \langle \nabla f(x), y - x \rangle | \leq \frac{L}{2}\|y - x\|^2. \quad (2.1)$$

3. Gradient-Based Methods for Noisy Quadratic Optimization

3.1. Inexact Finite Difference Approximations of the Gradient

The two standard types of the finite difference approximation of the gradient of the function $f : \mathbb{R}^n \to \mathbb{R}$ at a point $x \in \mathbb{R}^n$ with a finite difference interval $\delta > 0$ are taken from Wright (2006) as follows:

- Forward finite difference:

$$G_f(x,\delta) := \frac{1}{\delta}\sum_{i=1}^{n}\big(f(x+\delta e_i) - f(x)\big)e_i. \quad (3.1)$$

- Central finite difference:

$$G_c(x,\delta) := \frac{1}{2\delta}\sum_{i=1}^{n}\big(f(x+\delta e_i) - f(x-\delta e_i)\big)e_i. \tag{3.2}$$

For the quadratic function f_q defined in (1.1), explicit formulas for the forward and central finite difference approximations of the gradient ∇f_q are presented in the following proposition.

Proposition 3.1. Let f_q be the quadratic function defined in (1.1). Then, for all $(x,\delta) \in \mathbb{R}^n \times (0,+\infty)$, we have

$$G_f(x,\delta) = \frac{1}{\delta}\sum_{i=1}^{n}\big(f_q(x+\delta e_i) - f_q(x)\big)e_i = \nabla f_q(x) + \frac{\delta}{2}\sum_{i=1}^{n}\langle Ae_i, e_i\rangle e_i, \tag{3.3}$$

$$G_c(x,\delta) = \frac{1}{2\delta}\sum_{i=1}^{n}\big(f_q(x+\delta e_i) - f_q(x-\delta e_i)\big)e_i = \nabla f_q(x). \tag{3.4}$$

Proof. For all $(x,\delta) \in \mathbb{R}^n \times (0,+\infty)$, we have

$$\begin{aligned}
f_q(x+\delta e_i) &= \frac{1}{2}\langle A(x+\delta e_i), x+\delta e_i\rangle + \langle b, x+\delta e_i\rangle + c \\
&= \frac{1}{2}\big(\langle Ax, x\rangle + 2\delta\langle Ax, e_i\rangle + \delta^2\langle Ae_i, e_i\rangle\big) + \langle b, x\rangle + \delta\langle b, e_i\rangle + c \\
&= \frac{1}{2}\langle Ax, x\rangle + \langle b, x\rangle + c + \delta\langle Ax, e_i\rangle + \delta\langle b, e_i\rangle + \frac{\delta^2}{2}\langle Ae_i, e_i\rangle \\
&= f_q(x) + \delta\langle Ax, e_i\rangle + \delta\langle b, e_i\rangle + \frac{\delta^2}{2}\langle Ae_i, e_i\rangle.
\end{aligned} \tag{3.5}$$

It leads to

$$f_q(x+\delta e_i) - f_q(x) = \delta\langle Ax, e_i\rangle + \delta\langle b, e_i\rangle + \frac{1}{2}\delta^2\langle Ae_i, e_i\rangle.$$

Therefore

$$\begin{aligned}
G_f(x,\delta) &= \frac{1}{\delta}\sum_{i=1}^{n}\big(f_q(x+\delta e_i) - f_q(x)\big)e_i \\
&= \sum_{i=1}^{n}\big(\langle Ax, e_i\rangle + \langle b, e_i\rangle\big)e_i + \sum_{i=1}^{n}\frac{1}{2}\delta\langle Ae_i, e_i\rangle e_i \\
&= Ax + b + \frac{\delta}{2}\sum_{i=1}^{n}\langle Ae_i, e_i\rangle e_i \\
&= \nabla f_q(x) + \frac{\delta}{2}\sum_{i=1}^{n}\langle Ae_i, e_i\rangle e_i.
\end{aligned}$$

On the other hand, we also have

$$\begin{aligned} f_q(x-\delta e_i) &= \frac{1}{2}\langle A(x-\delta e_i), x-\delta e_i\rangle + \langle b, x-\delta e_i\rangle + c \\ &= \frac{1}{2}\Big(\langle Ax, x\rangle - 2\delta\langle Ax, e_i\rangle + \delta^2\langle Ae_i, e_i\rangle\Big) + \langle b, x\rangle - \delta\langle b, e_i\rangle + c \end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6) gives $f(x+\delta e_i) - f(x-\delta e_i) = 2\delta\langle Ax, e_i\rangle + 2\delta\langle b, e_i\rangle$ or

$\frac{1}{2\delta}(f(x+\delta e_i) - f(x-\delta e_i)) = \langle Ax, e_i\rangle + \langle b, e_i\rangle.$

This leads to

$$\begin{aligned} G_c(x,\delta) &= \frac{1}{2\delta}\sum_{i=1}^{n}\Big(f_q(x+\delta e_i) - f_q(x-\delta e_i)\Big)e_i \\ &= \sum_{i=1}^{n}\big(\langle Ax, e_i\rangle + \langle b, e_i\rangle\big)e_i \\ &= Ax + b = \nabla f_q(x), \end{aligned}$$

which completes the proof of Proposition 3.1.

Remark 3.2. *For all* $x \in \mathbb{R}^n$, *when the gradient* $\nabla f_q(x)$ *is approximated by the central finite difference scheme given in* (3.2), *the resulting approximation* $G_c(x,\delta)$ *is exact* $\nabla f_q(x)$ *for all* $\delta > 0$.

In the case where only noisy evaluations ϕ of the function f are available, we define the inexact finite difference approximation of the gradient of the function f at a point $x \in \mathbb{R}^n$ with a finite difference interval $\delta > 0$ as follows:

- Inexact forward finite difference:

$$\tilde{G}_f(x,\delta) := \frac{1}{\delta}\sum_{i=1}^{n}\big(\phi(x+\delta e_i) - \phi(x)\big)e_i. \tag{3.7}$$

- Inexact central finite difference:

$$\tilde{G}_c(x,\delta) := \frac{1}{2\delta}\sum_{i=1}^{n}\big(\phi(x+\delta e_i) - \phi(x-\delta e_i)\big)e_i. \tag{3.8}$$

Particularly, when $\phi(x) = f(x) + \xi(x)$ for all $x \in \mathbb{R}^n$,and there is a constant $\epsilon_f > 0$ such that

$|f(x) - \phi(x)| = |\epsilon(x)| \le \epsilon_f$ for all $x \in \mathbb{R}^n$,

the results on the error bounds between the inexact finite difference approximation to the gradient and the gradient of f is given in Berahas et al. (2022) as follows:

Theorem 3.2.

i. *Let* $f:\mathbb{R}^n \to \mathbb{R}$ *be continuously differentiable, and the gradient of* f *is Lipschitz continuous with the Lipschitz constant* L. *Let* $\tilde{G}_f(x,\delta)$ *denote the inexact forward finite difference approximation to the gradient* $\nabla f(x)$. *Then, for all* $x \in \mathbb{R}^n$ *and* $\delta > 0$,

$$\left\| \tilde{G}_f(x,\delta) - \nabla f(x) \right\| \le \frac{L\sqrt{n}\delta}{2} + \frac{2\sqrt{n}\xi_f}{\delta}. \quad (3.9)$$

ii. *Let* $f:\mathbb{R}^n \to \mathbb{R}$ *be twice continuously differentiable, and the Hessian of* f *is Lipschitz continuous with the Lipschitz constant* M. *Let* $\tilde{G}_c(x,\delta)$ *denote the inexact central finite difference approximation to the gradient* $\nabla f(x)$. *Then, for all* $x \in \mathbb{R}^n$ *and* $\delta > 0$,

$$\left\| \tilde{G}_c(x,\delta) - \nabla f(x) \right\| \le \frac{\sqrt{n}M\delta^2}{6} + \frac{\sqrt{n}\xi_f}{\delta}. \quad (3.10)$$

Combining Theorem 3.2 and Proposition 2.2 gives the following corollary about the error bounds between the inexact finite difference approximation to the gradient and the gradient of the quadratic function f_q defined in (1.1).

Corollary 3.3. *Let* f_q *be the quadratic function defined in (1.1). The error bounds between the inexact finite difference approximation to the gradient and the gradient of* f *is given by*

$$\left\| \tilde{G}_f(x,\delta) - \nabla f_q(x) \right\| \le \frac{\|A\|\sqrt{n}\delta}{2} + \frac{2\sqrt{n}\xi_f}{\delta}, \quad (3.11)$$

and

$$\left\| \tilde{G}_c(x,\delta) - \nabla f_q(x) \right\| \le \frac{\sqrt{n}\xi_f}{\delta}. \quad (3.12)$$

For the quadratic function f_q, the error bound for the inexact central finite difference approximation given in equation (3.12) is strictly smaller than that of the forward finite difference in equation (3.11). This improvement arises because the Hessian of a quadratic function is Lipschitz continuous with a Lipschitz constant $M = 0$. Motivated by this observation, we employ the inexact gradient descent with fixed step size using the inexact central finite difference scheme.

3.2. Convergence Analysis with Inexact Gradient Descent using Central Difference Scheme

For the quadratic function f_q defined in (1.1), as A is symmetric and positive definite, the unique global minimizer is given by $x^* = -A^{-1}b,$ and the corresponding minimum value is denoted by

$$f_q^* := f_q(x^*) = \min_{x\in\mathbb{R}^n} f_q(x).$$

From (1.5) and lemma (2.1), for the algorithm described in (1.4), the following theorem provides an estimate of the distance between $f_q(x^k)$ $f_q(x^k)$ and the optimal solution f^* after k iterations.

Theorem 3.5. *Let* $\{x^k\}$ *be the sequence generated by the Algorithm* (1.4). *If we run the Algorithm* (1.4) *for* k *iterations with a fixed step size* $0 < t < \frac{2}{\|A\|}$, *we derive the estimate*

$$f_q(x^k) - f_q^* \le \frac{\|A\|}{2}\left(\rho^k\left(\|x^0 - x^*\| - \frac{t\gamma}{1-\rho}\right) + \frac{t\gamma}{1-\rho}\right)^2,$$

where $\rho = \|I - tA\| < 1$.

Proof. As $f_q(x^*) = f_q^*$ and $\nabla f_q(x^*) = 0$, it follows from (1.5) and (2.1) that

$$\begin{aligned} f_q(x^k) - f_q^* &\le \frac{\|A\|}{2}\|x^k - x^*\|^2 \\ &\le \frac{\|A\|}{2}\left(\rho^k\left(\|x^0 - x^*\| - \frac{t\gamma}{1-\rho}\right) + \frac{t\gamma}{1-\rho}\right)^2. \end{aligned}$$

To solve the problem given in *(1.1)*, we consider the following algorithm called Inexact gradient descent using central difference scheme (IGDC), which is the inexact gradient descent method with a fixed step size described in (1.4) using an inexact central finite difference scheme as an approximation to the gradient of f_q.

Algorithm 1 (IGDC).

Step 0 (initialization). Choose an initial point $x^0 \in \mathbb{R}^n$, $\delta > 0$, $t > 0$. Set $k := 0$ and $g^k = \tilde{G}_c(x^k, \delta)$.

Step 1 (update).

$$x^{k+1} = x^k - t \cdot g^k. \tag{3.13}$$

We consider two special cases of the noisy approximation ϕ_q using the Algorithm (IGDC).

3.2.1. Convergence Analysis with Inexact Gradient Descent using Central Difference Scheme for Type I

Proposition 3.6. *With the setting of Type I, the error bound between the inexact central difference approximation to the gradient and the gradient of* f_q *is given by*

$$\left\|\tilde{G}_c(x,\delta) - \nabla f_q(x)\right\| \le \sqrt{n} \cdot \overline{\xi_b}\left(\frac{\|x\|}{\delta} + 1\right) + \frac{\sqrt{n} \cdot \overline{\xi_c}}{\delta} \quad \textit{for any } (x,\delta) \in \mathbb{R}^n \times (0,\infty). \tag{3.14}$$

Let $(x,\delta) \in \mathbb{R}^n \times (0,\infty)$. Note that

$$\begin{aligned}\phi_q(x) &= \frac{1}{2}\langle Ax, x\rangle + \langle b + \xi_b(x), x\rangle + c + \xi_c(x)\\ &= \left(\frac{1}{2}\langle Ax, x\rangle + \langle b, x\rangle + c\right) + \langle \xi_b(x), x\rangle + \xi_c(x)\\ &= f_q(x) + \langle \xi_b(x), x\rangle + \xi_c(x).\end{aligned}$$

It follows from (3.4) and (3.8) that

$$\begin{aligned}\tilde{G}_c(x,\delta) &= \frac{1}{2\delta}\sum_{i=1}^{n}\left[f_q(x+\delta e_i) - f_q(x-\delta e_i)\right]e_i\\ &\quad + \frac{1}{2\delta}\sum_{i=1}^{n}\left(\langle \xi_b(x+\delta e_i), x+\delta e_i\rangle - \langle \xi_b(x-\delta e_i), x-\delta e_i\rangle\right)e_i\\ &\quad + \frac{1}{2\delta}\sum_{i=1}^{n}\left(\xi_c(x+\delta e_i) - \xi_c(x-\delta e_i)\right)e_i\\ &= G_c(x,\delta) + \xi_b(x,\delta) + \xi_c(x,\delta)\\ &= \nabla f_q(x) + \xi_b(x,\delta) + \xi_c(x,\delta).\end{aligned}$$

where

- $\xi_b(x,\delta) = \frac{1}{2\delta}\sum_{i=1}^{n}\left(\langle \xi_b(x+\delta e_i), x+\delta e_i\rangle - \langle \xi_b(x-\delta e_i), x-\delta e_i\rangle\right)e_i$ and
- $\xi_c(x,\delta) = \frac{1}{2\delta}\sum_{i=1}^{n}\left(\xi_c(x+\delta e_i) - \xi_c(x-\delta e_i)\right)e_i$.

On the other hand, we have

$$\langle \xi_b(x+\delta e_i), x+\delta e_i\rangle = \langle \xi_b(x+\delta e_i), x\rangle + \delta\langle \xi_b(x+\delta e_i), e_i\rangle,$$
$$\langle \xi_b(x-\delta e_i), x-\delta e_i\rangle = \langle \xi_b(x-\delta e_i), x\rangle - \delta\langle \xi_b(x-\delta e_i), e_i\rangle.$$

Thus,

$$\langle \xi_b(x+\delta e_i), x+\delta e_i\rangle - \langle \xi_b(x-\delta e_i), x-\delta e_i\rangle = \langle \xi_b(x+\delta e_i) - \xi_b(x-\delta e_i), x\rangle + \delta\langle \xi_b(x+\delta e_i) + \xi_b(x-\delta e_i), e_i\rangle.$$

This implies

$$\begin{aligned}\|\xi_b(x,\delta)\| &= \frac{1}{2\delta}\left\|\sum_{i=1}^{n}\left(\langle \xi_b(x+\delta e_i) - \xi_b(x-\delta e_i), x\rangle + \delta\langle \xi_b(x+\delta e_i) + \xi_b(x-\delta e_i), e_i\rangle\right)e_i\right\|\\ &= \frac{1}{2\delta}\sqrt{\sum_{i=1}^{n}\left(\langle \xi_b(x+\delta e_i) - \xi_b(x-\delta e_i), x\rangle + \delta\langle \xi_b(x+\delta e_i) + \xi_b(x-\delta e_i), e_i\rangle\right)^2}\\ &\le \frac{1}{2\delta}\sqrt{\sum_{i=1}^{n}\left(\|\xi_b(x+\delta e_i) - \xi_b(x-\delta e_i)\|\cdot\|x\| + \delta\|\xi_b(x+\delta e_i) + \xi_b(x-\delta e_i)\|\right)^2}\\ &\le \frac{1}{2\delta}\sqrt{\sum_{i=1}^{n}\left(2\cdot\overline{\xi_b}\,\|x\| + \delta\cdot 2\cdot\overline{\xi_b}\right)^2}\\ &= \sqrt{n}\cdot\overline{\xi_b}\left(\frac{\|x\|}{\delta} + 1\right).\end{aligned}$$

Moreover,

$$\begin{aligned}\|\xi_c(x,\delta)\| &= \left\|\frac{1}{2\delta}\sum_{i=1}^{n}\big(\xi_c(x+\delta e_i)-\xi_c(x-\delta e_i)\big)e_i\right\| \\ &= \frac{1}{2\delta}\sqrt{\sum_{i=1}^{n}\big(\xi_c(x+\delta e_i)-\xi_c(x-\delta e_i)\big)^2} \\ &\le \frac{1}{2\delta}\sqrt{\sum_{i=1}^{n}(2\cdot\overline{\xi_c})^2} \\ &\le \frac{\sqrt{n}\cdot\overline{\xi_c}}{\delta}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\tilde{G}_c(x,\delta)-\nabla f(x)\| &\le \|\xi_b(x,\delta)\|+\|\xi_c(x,\delta)\| \\ &\le \sqrt{n}\cdot\overline{\xi_b}\left(\frac{\|x\|}{\delta}+1\right)+\frac{\sqrt{n}\cdot\overline{\xi_c}}{\delta}.\end{aligned}$$

This completes the proof of Proposition 3.6.

Theorem 3.7. If we run Algorithm (IGDC) with a fixed step size t such that

$$0<t<\min\left\{\frac{2}{\|A\|},\ \frac{(1-\rho)\|x^0\|}{\theta+\sqrt{n}\cdot\overline{\xi_b}\left(\frac{\|x^0\|}{\delta}+1\right)+\frac{\sqrt{n}\cdot\overline{\xi_c}}{\delta}}\right\},$$

it will yield a sequence of $\{x^k\}$ that satisfies

$$\|x^k\|\le\|x^0\| \text{ for all } k\in\mathbb{N}. \tag{3.15}$$

Consequently, for all $k\in\mathbb{N}$, the gradient approximation error satisfies:

$$\|\tilde{G}_c(x^k,\delta)-\nabla f_q(x^k)\|\le\sqrt{n}\cdot\overline{\xi_b}\left(\frac{\|x^0\|}{\delta}+1\right)+\frac{\sqrt{n}\cdot\overline{\xi_c}}{\delta}. \tag{3.16}$$

Therefore, after k iterations, we obtain the following estimates:

$$\|x^k-x^*\|\le\frac{t\gamma}{1-\rho}+\rho^k\left(\|x^0-x^*\|-\frac{t\gamma}{1-\rho}\right),$$

and

$$f_q(x^k)-f_q^*\le\frac{\|A\|}{2}\left(\rho^k\left(\|x^0-x^*\|-\frac{t\gamma}{1-\rho}\right)+\frac{t\gamma}{1-\rho}\right)^2,$$

where $\rho=\|I-tA\|<1$ and $\gamma=\sqrt{n}\cdot\overline{\xi_b}\left(\frac{\|x^0\|}{\delta}+1\right)+\frac{\sqrt{n}\cdot\overline{\xi_c}}{\delta}$.

Proof. We will prove (3.15) by induction on k.

For $k=0$, the claim (3.15) clearly holds as

$$\|x^0\| = \|x^0\|.$$

Assume that $\|x^k\| \le \|x^0\|$ for some $k \in \mathbb{N}$. We must show that $\|x^{k+1}\| \le \|x^0\|$. From the update rule $x^{k+1} = x^k - t\tilde{G}_c(x^k, \delta)$, $\|x^k\| \le \|x^0\|$ and the bound (3.14), using the triangle inequality gives

$$\begin{aligned}
\|x^{k+1}\| &\le \|x^k - t\nabla f_q(x^k)\| + t\|\tilde{G}_c(x^k,\delta) - \nabla f_q(x^k)\| \\
&\le \|x^k - t(Ax^k + b)\| + t\left(\sqrt{n}\cdot\overline{\xi_b}\left(\frac{\|x^k\|}{\delta}+1\right) + \frac{\sqrt{n}\cdot\overline{\xi_c}}{\delta}\right) \\
&\le \|I - tA\|\cdot\|x^k\| + t\|b\| + t\left(\sqrt{n}\cdot\overline{\xi_b}\left(\frac{\|x^k\|}{\delta}+1\right) + \frac{\sqrt{n}\cdot\overline{\xi_c}}{\delta}\right) \\
&\le \rho\cdot\|x^0\| + t\theta + t\left(\sqrt{n}\cdot\overline{\xi_b}\left(\frac{\|x^0\|}{\delta}+1\right) + \frac{\sqrt{n}\cdot\overline{\xi_c}}{\delta}\right).
\end{aligned}$$

As $t \le \dfrac{(1-\rho)\|x^0\|}{\theta + \sqrt{n}\cdot\overline{\xi_b}\left(\dfrac{\|x^0\|}{\delta}+1\right) + \dfrac{\sqrt{n}\cdot\overline{\xi_c}}{\delta}}$, then

$$\|x^{k+1}\| \le \|x^0\|.$$

Hence,

$$\|x^k\| \le \|x^0\| \text{ for all } k \in \mathbb{N}.$$

It also follows from (3.14)

$$\|\tilde{G}_c(x^k,\delta) - \nabla f_q(x^k)\| \le \sqrt{n}\cdot\overline{\xi_b}\left(\frac{\|x^0\|}{\delta}+1\right) + \frac{\sqrt{n}\cdot\overline{\xi_c}}{\delta} \text{ for all } k \in \mathbb{N}.$$

Applying (1.5) and Theorem 3.5 for $\gamma = \sqrt{n}\cdot\overline{\xi_b}\left(\dfrac{\|x^0\|}{\delta}+1\right) + \dfrac{\sqrt{n}\cdot\overline{\xi_c}}{\delta}$ gives the following estimates

$$\|x^k - x^*\| \le \frac{t\gamma}{1-\rho} + \rho^k\left(\|x^0 - x^*\| - \frac{t\gamma}{1-\rho}\right),$$

and

$$f_q(x^k) - f_q^* \le \frac{\|A\|}{2}\left(\rho^k\left(\|x^0 - x^*\| - \frac{t\gamma}{1-\rho}\right) + \frac{t\gamma}{1-\rho}\right)^2.$$

This completes the proof of Theorem 3.7.

3.2.2. Convergence Analysis with Inexact Gradient Descent using Central Difference Scheme for Type II

Note that the model in type (1.3) is a special case of the model in type (1.2) with $\xi_b(x) = 0$ for all $x \in \mathbb{R}^n$ and $\overline{\xi_b} = 0$. Applying the error bound in (3.16) yields the following proposition.

Proposition 3.8. *With the setting in Type II, the error bound between the inexact central difference approximation to the gradient and the gradient of f is given by*

$$\left\|\tilde{G}_c(x,\delta) - \nabla f_q(x)\right\| \le \frac{\sqrt{n}\cdot\overline{\xi_c}}{\delta} \quad \text{for any } (x,\delta) \in \mathbb{R}^n \times (0,\infty).$$

Applying Theorem 3.5 and (1.5) for $\gamma = \dfrac{\sqrt{n}\cdot\overline{\xi_c}}{\delta}$ gives the following theorem.

Theorem 3.9. *Let $\{x^k\}$ be the sequence generated by Algorithm IGDC. If we run Algorithm IGDC for k iterations with a fixed step size $0 < t < \dfrac{2}{\|A\|}$, we obtain the following estimates:*

$$\left\|x^k - x^*\right\| \le \frac{t\gamma}{1-\rho} + \rho^k\left(\|x^0 - x^*\| - \frac{t\gamma}{1-\rho}\right),$$

and

$$f_q(x^k) - f_q^* \le \frac{\|A\|}{2}\left(\rho^k\left(\|x^0 - x^*\| - \frac{t\gamma}{1-\rho}\right) + \frac{t\gamma}{1-\rho}\right)^2,$$

where $\rho = \|I - tA\|$ and $\gamma = \dfrac{\sqrt{n}\cdot\overline{\xi_c}}{\delta}$.

Remark 3.10. *The main difference between Type I and Type II lies in the error bound between the approximation of the gradient using the central finite difference scheme and the true gradient of f_q. Specifically, in Type I, this error bound is not bounded for all $x \in \mathbb{R}^n$. Therefore, in the Type I setting, it is necessary to know an upper bound of $\|b\|$ to adjust the step size t properly, ensuring that the error between the finite difference approximation and the true gradient of f_q remains bounded for all iterates x^k generated by Algorithm (IGDC).*

In contrast, for Type II, the error bound between the approximation of the gradient and the true gradient of f_q is bounded for all $x \in \mathbb{R}^n$. As a result, the step size t in this case only needs to satisfy the condition $0 < t < \dfrac{2}{\|A\|}$.

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CÁC PHƯƠNG PHÁP DỰA TRÊN GRADIENT CHO BÀI TOÁN TỐI ƯU HÓA BẬC HAI CÓ NHIỄU

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TÓM TẮT

Trong bài báo này, chúng tôi nghiên cứu bài toán tối ưu một hàm bậc hai nhiều biến có nhiễu bằng cách sử dụng phương pháp gradient xấp xỉ, trong đó đạo hàm được ước lượng thông qua các lược đồ sai phân trung tâm. Hai mô hình nhiễu được phân tích: (i) ma trận bậc hai được biết, trong khi các thành phần tuyến tính và hằng số bị nhiễu; (ii) cả ma trận và thành phần tuyến tính đều được biết, chỉ có thành phần hằng số là nhiễu. Chúng tôi xây dựng các chặn sai số hiển, thiết lập tốc độ hội tụ dựa trên các ước lượng của Polyak, và xác định độ phức tạp cho dãy lặp sinh bởi phương pháp gradient xấp xỉ với bước nhảy cố định.

Từ khóa: phương pháp sai phân trung tâm; tối ưu hóa không dùng đạo hàm; thuật toán gradient xấp xỉ; hàm bậc hai nhiều biến có nhiễu