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Research Article

A NOTE ON DECOMPOSITIONS OF MATRICES IN THE CENTRALIZER OF A WEYR MATRIX

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ABSTRACT

Let $\mathbb{F}$ be an algebraically closed field of characteristic not equal to 2, and n be a positive integer. Assume $S \in \mathrm{M}_{3n}(\mathbb{F})$ is a basic Weyr matrix with a homogeneous Weyr structure of length 3, and $\mathcal{C}(S)$ denotes the centralizer of S in $\mathrm{M}_{3n}(\mathbb{F})$. This paper describes the subgroup generated by commutators of involutions in $\mathcal{C}(S)$. Furthermore, we show that every matrix in this subgroup can be expressed as a product of at most $2n^2$ commutators of involutions in $\mathcal{C}(S)$.

Keywords: commutator; involution; matrix centralizer; matrix decomposition; Weyr matrix

1. Introduction

1.1. Problems

Let $\mathbb{F}$ be a field, n be a positive integer, and $\mathrm{M}_n(\mathbb{F})$ be the ring of $n \times n$ square matrices over $\mathbb{F}$. Denote by $\mathrm{GL}_n(\mathbb{F})$ and $\mathrm{SL}_n(\mathbb{F})$ the general linear group and the special linear group of degree n over $\mathbb{F}$, respectively. The zero matrix and the identity matrix in $\mathrm{M}_n(\mathbb{F})$ are denoted by 0_n and I_n (or simply 0 and I). For each matrix $A \in \mathrm{M}_n(\mathbb{F})$, let $\mathrm{tr}(A)$ denote its trace. A matrix $A \in \mathrm{GL}_n(\mathbb{F})$ is called an involution if it satisfies $A^2 = \mathrm{I}_n$. A matrix $A \in \mathrm{M}_n(\mathbb{F})$ is called a scalar matrix if there exists $\lambda \in \mathbb{F}$ such that $A = \lambda \mathrm{I}_n$.

Determining necessary and sufficient conditions for a square matrix to be expressed as a product of involutions has attracted considerable interest. Wonenburger (1966) proved that if $\mathbb{F}$ is a field of characteristic not equal to 2, then a square matrix A over $\mathbb{F}$ is a product of two involutions if and only if A is invertible and similar to its inverse A^{-1}. A year later, Djoković (1967) extended this result to fields of arbitrary characteristic.

Denote by $[A, B] = ABA^{-1}B^{-1}$ the commutator of two invertible matrices A and B. The decomposition of a square matrix as a product of commutators of involutions has also

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attracted significant attention from mathematicians. Zheng (2002) proved that if $\mathbb{F}=\mathbb{R}$ or $\mathbb{F}=\mathbb{C}$, then every matrix in $\mathrm{SL}_n(\mathbb{F})$ is a product of at most two commutators of involutions in $\mathrm{GL}_n(\mathbb{F})$. Hou (2019) extended this result to fields of characteristic not equal to 2. The result is stated as follows.

Theorem 1.1. *(Hou, 2019, Theorem 2.8) Assume that $\mathbb{F}$ is an arbitrary field of characteristic not equal to 2. Then every element of the group $\mathrm{SL}_n(\mathbb{F})$ can be written as a product of at most two commutators of involutions in $\mathrm{GL}_n(\mathbb{F})$.*

More recently, Son et al. (2022) showed that if $\mathbb{F}$ is a field with at least three elements and $n>1$, then every matrix in $\mathrm{SL}_n(\mathbb{F})$ is a product of at most two commutators of involutions.

In this context, decomposing a square matrix as a product of involutions satisfying certain conditions is often challenging. For example, given a matrix S, the problem of decomposing a square matrix as a product of involutions that commute with S was raised in a recent paper by De La Cruz and Tañedo (2022).

Let S be a square matrix of degree n over a field $\mathbb{F}$. The centralizer of S in $\mathrm{M}_n(\mathbb{F})$, denoted by $\mathcal{C}(S)$, is the set of all $n\times n$ matrices over $\mathbb{F}$ that commute with S, i.e.,

$$\mathcal{C}(S)=\{A\in \mathrm{M}_n(\mathbb{F}) \mid AS=SA\}.$$

Definition 1.2. *(O'Meara et al., 2011, Definition 2.1.1) A basic Weyr matrix with eigenvalue λ is an n-by-n matrix W of the following form: There is a partition $n=n_1+n_2+\cdots+n_r$ of n with $n_1\geq n_2\geq\ldots\geq n_r\geq 1$ such that, when W is viewed as an r-by-r block matrix (W_{ij}), where the (i,j) block W_{ij} is an n_i-by-n_j matrix, the following three properties hold:*

(1) The main diagonal blocks W_{ii} are the n_i-by-n_i scalar matrices $\lambda \mathrm{I}_{n_i}$ for $i=1,2,\ldots,r$.

(2) The first superdiagonal blocks $W_{i,i+1}$ are full-column-rank n_i-by-n_{i+1} matrices in reduced row-echelon form (that is, an identity matrix followed by zero rows) for $i=1,2,\ldots,r-1$.

(3) All other blocks of W are zero; that is, $W_{ij}=0$ for $j\notin\{i,i+1\}$.

In this case, we say that W has Weyr structure $(n_1,n_2,\ldots,n_r)$ of length r.

Let W be a basic Weyr matrix with Weyr structure $(n_1,n_2,\ldots,n_r)$. Then, W is said to have a homogeneous Weyr structure if $n_1=n_2=\cdots=n_r$. Otherwise, W has a nonhomogeneous Weyr structure.

In Section 2, we provide a detailed description of Weyr matrices, and additional background can be found in O'Meara et al. (2011). Suppose S is a basic Weyr matrix with a homogeneous Weyr structure of length at most 3. De La Cruz and Tañedo (2022) established necessary and sufficient conditions for a matrix to be a product of involutions in $\mathcal{C}(S)$. Their result for the case of length 3, which is central to our paper, is as follows.

Theorem 1.3. (De La Cruz & Tañedo, 2022, *Theorem 1.1.b*) *Let S be a basic Weyr matrix with a homogeneous Weyr structure of length 3. A matrix X is a product of involutions in $\mathcal{C}(S)$ if and only if*

$$X = \begin{pmatrix} X_1 & X_2 & X_3 \\ & X_1 & X_2 \\ & & X_1 \end{pmatrix},$$

with $\det X_1 = \pm 1$, $\operatorname{tr}\left(X_1^{-1}X_2\right) = 0$ *and* $2\operatorname{tr}\left(X_1^{-1}X_3\right) = \operatorname{tr}\left(\left(X_1^{-1}X_2\right)^2\right)$.

In this paper, we seek necessary and sufficient conditions for a matrix to be a product of commutators of involutions in $\mathcal{C}(S)$. Consequently, we describe the subgroup generated by the commutators of involutions in $\mathcal{C}(S)$.

Let $\mathbb{F}$ be an algebraically closed field of characteristic different from 2, and n be a positive integer. Suppose $S \in \mathrm{M}_{3n}(\mathbb{F})$ is a basic Weyr matrix with a homogeneous Weyr structure of length 3. Let $\mathcal{C}(S)$ be the centralizer of S in $\mathrm{M}_{3n}(\mathbb{F})$. Denote $\mathcal{I}_{\mathcal{C}(S)}$ as the subgroup generated by the commutators of involutions in $\mathcal{C}(S)$. In this paper, we address the following two problems.

Problem 1.4. *Describe all matrices in $\mathcal{I}_{\mathcal{C}(S)}$.*

For each element $X \in \mathcal{I}_{\mathcal{C}(S)} \setminus \{\mathrm{I}\}$, we define $\ell(X)$ to be the smallest positive integer n such that X can be written as a product of n commutators of involutions in $\mathcal{C}(S)$. By convention, $\ell(\mathrm{I}) = 0$. The width of the group $\mathcal{I}_{\mathcal{C}(S)}$, denoted $\omega\left(\mathcal{I}_{\mathcal{C}(S)}\right)$, is the least upper bound of the set $\left\{\ell(X) \mid X \in \mathcal{I}_{\mathcal{C}(S)}\right\}$.

Problem 1.5. *Estimate the width $\omega\left(\mathcal{I}_{\mathcal{C}(S)}\right)$ of the group $\mathcal{I}_{\mathcal{C}(S)}$.*

1.2. Main result

The main result of this paper is stated in the following theorem.

Theorem 1.6. *Let $\mathbb{F}$ be an algebraically closed field of characteristic different from 2, and n be a positive integer. Suppose $S \in \mathrm{M}_{3n}(\mathbb{F})$ is a basic Weyr matrix with a homogeneous Weyr structure of length 3. Then, the subgroup $\mathcal{I}_{\mathcal{C}(S)}$ consists of all matrices X of the form*

$$X = \begin{pmatrix} X_1 & X_2 & X_3 \\ & X_1 & X_2 \\ & & X_1 \end{pmatrix},$$

where $X_1 \in \mathrm{SL}_n(\mathbb{F})$ *and* $X_2, X_3 \in \mathrm{M}_n(\mathbb{F})$ *such that* $\operatorname{tr}\left(X_1^{-1}X_2\right) = 0$ *and* $2\operatorname{tr}\left(X_1^{-1}X_3\right) = \operatorname{tr}\left(\left(X_1^{-1}X_2\right)^2\right)$. *Furthermore,* $\omega\left(\mathcal{I}_{\mathcal{C}(S)}\right) \leq 2n^2$.

2. Weyr Matrices

In this section, we recall some basic facts about Weyr matrices and their centralizers. The remarks in this section are straightforward to verify, so their proofs are omitted.

Throughout this paper, we denote by $\mathbb{F}$ an algebraically closed field of characteristic not equal to 2. The direct sum of matrices $A_i \in \mathrm{M}_{n_i}(\mathbb{F})$ for $i \in \overline{1,k}$, denoted by $\bigoplus_{i=1}^{k} A_i$, is the block-diagonal matrix of the form

$$\bigoplus_{i=1}^{k} A_i = \begin{pmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & A_k \end{pmatrix} \in \mathrm{M}_n(\mathbb{F}),$$

where $n = n_1 + n_2 + \cdots + n_k$.

Remark 2.1. *Let $n_1, n_2, ..., n_r, k$ be positive integers. If A_i is a product of k commutators of involutions in $\mathrm{M}_{n_i}(\mathbb{F})$ for every $i \in \overline{1,r}$, then $\bigoplus_{i=1}^{r} A_i$ is a product of k commutators of involutions in $\mathrm{M}_n(\mathbb{F})$, where $n = n_1 + n_2 + \cdots + n_r$.*

Remark 2.2. *Let $S \in \mathrm{M}_n(\mathbb{F})$ and $A \in \mathcal{C}(S)$. If A is a product of k commutators of involutions in $\mathcal{C}(S)$, then A^{-1} is also a product of k commutators of involutions in $\mathcal{C}(S)$.*

Remark 2.3. *Suppose S_1 and S_2 are matrices in $\mathrm{M}_n(\mathbb{F})$ such that $S_2 = X^{-1}S_1X$, for some $X \in \mathrm{GL}_n(\mathbb{F})$. Let $A \in \mathrm{M}_n(\mathbb{F})$ and $B = X^{-1}AX$. Then, A is a product of k commutators of involutions in $\mathcal{C}(S_1)$ if and only if B is a product of k commutators of involutions in $\mathcal{C}(S_2)$.*

Remark 2.3 indicates that we can study products of commutators of involutions in the centralizer of a canonical form (under similarity) rather than in the centralizer of an arbitrary matrix. In practice, studying the centralizer of an arbitrary matrix is challenging, especially when attempting to describe the involutions it contains. For instance, De La Cruz and Tañedo (2022) point out that using the Jordan canonical form of a matrix to study products of involutions in $\mathcal{C}(S)$, where S is an arbitrary matrix, is inefficient. Consequently, studying products of commutators of involutions also becomes difficult when relying on the Jordan form. By contrast, using the Weyr canonical form allows for a more effective description of involutions in the centralizer, leading to several results. Below, we present some fundamental facts about this canonical form.

Definition 2.4. *(O'Meara et al., 2011, Definition 2.1.5) Let W be a square matrix over an algebraically closed field $\mathbb{F}$, and let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct eigenvalues of W. We say that W is in Weyr form (or is a Weyr matrix)* if W is a direct sum of basic Weyr matrices, one for each distinct eigenvalue. In other words, W has the form

$$W = \bigoplus_{i=1}^{k} W_i = \begin{pmatrix} W_1 & & & \\ & W_2 & & \\ & & \ddots & \\ & & & W_k \end{pmatrix},$$

where W_i is a basic Weyr matrix with eigenvalue λ_i for $i \in \overline{1,k}$.

Let $S \in \mathrm{M}_n(\mathbb{F})$. By Theorem 2.2.4 of O'Meara et al. (2011), there exists a Weyr matrix W such that S is similar to W, that is, $W = X^{-1}SX$ for some $X \in \mathrm{GL}_n(\mathbb{F})$. According to Remark 2.3, a matrix A in $\mathcal{C}(S)$ is a product of commutators of involutions in $\mathcal{C}(S)$ if and only if $X^{-1}AX$ is a product of commutators of involutions in $\mathcal{C}(W)$. This observation allows us to use the Weyr canonical form when studying products of commutators of involutions in the centralizer of an arbitrary matrix.

Remark 2.5. *Let $\lambda_1, \lambda_2, \ldots, \lambda_r \in \mathbb{F}$ be distinct. Suppose that $W = \bigoplus_{i=1}^{r} W_i$ is a Weyr matrix in which W_i is a basic Weyr matrix corresponding to the eigenvalue λ_i for all $i \in \overline{1,r}$. A matrix $X \in \mathcal{C}(W)$ is a product of k commutators of involutions in $\mathcal{C}(W)$ if and only if there exists a decomposition $X = \bigoplus_{i=1}^{r} X_i$, where $X_i \in \mathcal{C}(W_i)$ and is a product of k commutators of involutions in $\mathcal{C}(W_i)$, for each $i \in \overline{1,r}$.*

From Remark 2.5, the study of products of commutators of involutions in the centralizer of a Weyr matrix can be reduced to the case of a basic Weyr matrix. In what follows, we apply Proposition 2.3.3 of O'Meara et al. (2011) to derive the following result.

Corollary 2.6. *Let n be a positive integer and let $S \in \mathrm{M}_{3n}(\mathbb{F})$ be a basic Weyr matrix with a homogeneous Weyr structure of length 3. Then, a matrix $X \in \mathrm{M}_{3n}(\mathbb{F})$ is in $\mathcal{C}(S)$ if and only if X has the form*

$$X = \begin{pmatrix} X_1 & X_2 & X_3 \\ & X_1 & X_2 \\ & & X_1 \end{pmatrix},$$

where $X_i \in \mathrm{M}_n(\mathbb{F})$, for each $i \in \{1,2,3\}$.

3. Proof of the main result

3.1. Notation. We introduce the matrix notation used in this paper. Let i, j and n be positive integers. We use the following notation:

- $E_{i,j}$ denotes an $n \times n$ matrix whose entry in row i and column j is 1, and all other entries are 0.
- $D_i^{(n)} = (d_{st})$ denotes an $n \times n$ diagonal matrix whose ith diagonal entry is -1, and all other diagonal entries are 1.
- R_n denotes an $n \times n$ matrix in which the entries on the secondary diagonal are all 1, and all other entries are 0.
- $J_i^{(n)} = \mathrm{I}_{i-1} \oplus R_2 \oplus \mathrm{I}_{n-i-1}$.

Direct computations and elementary transformations yield the following two propositions.

Proposition 3.1. For any positive integers i, j, n and the notation introduced above, we have:

(1) $\left(D_i^{(n)}\right)^2 = \left(R_n\right)^2 = \left(J_i^{(n)}\right)^2 = \mathrm{I}_n$;

(2) $E_{i,j} = \begin{cases} -E_{i,i}D_i^{(n)} = -D_i^{(n)}E_{i,i} & \text{if } i = j, \\ E_{i,j}D_i^{(n)} = -D_i^{(n)}E_{i,j} & \text{if } i \neq j; \end{cases}$

(3) $\left(E_{i,j}\right)^2 = \begin{cases} E_{i,i} & \text{if } i = j, \\ 0_n & \text{if } i \neq j; \end{cases}$

(4) $J_i^{(n)}\left(E_{i+1,i} - E_{i,i+1}\right) + \left(E_{i+1,i} - E_{i,i+1}\right)J_i^{(n)} = 0_n$;

(5) $J_i^{(n)}E_{i+1,i} + \left(E_{i+1,i} - E_{i,i+1}\right)^2 + E_{i+1,i}\,J_i^{(n)} = 0_n$.

Proposition 3.2. *Let n be a positive integer and $A = \left(a_{ij}\right) \in \mathrm{M}_n\left(\mathbb{F}\right)$. For each positive integer $s \in \{1, 2, \ldots, n\}$, we define $a_s = \sum_{i=1}^{s} a_{ii}$. Then, we have*

$$A + \sum_{i=1}^{n-1} a_i\left(E_{i+1,i+1} - E_{i,i}\right) = \sum_{1 \leq i \neq j \leq n} a_{ij}E_{i,j} + a_nE_{n,n}.$$

3.2. Proof of Theorem 1.6. Some of the propositions and lemmas could be stated for basic Weyr matrices with a homogeneous Weyr structure of length greater than 3. However, since this extension is not essential for proving Theorem 1.6, we restrict the results to the case of length 3 to simplify the computations.

Proposition 3.3. *Let $S \in \mathrm{M}_{3n}\left(\mathbb{F}\right)$ be a basic Weyr matrix with a homogeneous Weyr structure of length 3 and $n \geq 2$. Then, for any $a \in \mathbb{F}$ and any $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$, the matrices*

$$X = \begin{pmatrix} D_i^{(n)} & 0_n & aE_{i,j} \\ & D_i^{(n)} & 0_n \\ & & D_i^{(n)} \end{pmatrix} \text{ and } Y = \begin{pmatrix} D_i^{(n)} & aE_{i,j} & 0_n \\ & D_i^{(n)} & aE_{i,j} \\ & & D_i^{(n)} \end{pmatrix}$$

are involutions in $\mathcal{C}(S)$.

Proof. By Corollary 2.6, X and Y are matrices in $\mathcal{C}(S)$. Furthermore, by direct calculation and Proposition 3.1, we have

$$X^2 = \begin{pmatrix} \left(D_i^{(n)}\right)^2 & 0_n & a\left(D_i^{(n)}E_{i,j} + E_{i,j}D_i^{(n)}\right) \\ & \left(D_i^{(n)}\right)^2 & 0_n \\ & & \left(D_i^{(n)}\right)^2 \end{pmatrix} = \begin{pmatrix} \mathrm{I}_n & 0_n & 0_n \\ & \mathrm{I}_n & 0_n \\ & & \mathrm{I}_n \end{pmatrix} = \mathrm{I}_{3n},$$

$$Y^2 = \begin{pmatrix} \left(D_i^{(n)}\right)^2 & a\left(D_i^{(n)}E_{i,j} + E_{i,j}D_i^{(n)}\right) & a^2\left(E_{i,j}\right)^2 \\ & \left(D_i^{(n)}\right)^2 & a\left(D_i^{(n)}E_{i,j} + E_{i,j}D_i^{(n)}\right) \\ & & \left(D_i^{(n)}\right)^2 \end{pmatrix} = \begin{pmatrix} \mathrm{I}_n & 0_n & 0_n \\ & \mathrm{I}_n & 0_n \\ & & \mathrm{I}_n \end{pmatrix} = \mathrm{I}_{3n}.$$

Therefore, X and Y are involutions in $\mathcal{C}(S)$.

Proposition 3.4. *Let $S \in \mathrm{M}_{3n}(\mathbb{F})$ be a basic Weyr matrix with a homogeneous Weyr structure of length 3, where $n \geq 2$. Then, for any $a \in \mathbb{F}$ and any $i \in \{1, 2, \ldots, n-1\}$, the matrices*

$$X = \begin{pmatrix} J_i^{(n)} & 0_n & a\left(E_{i+1,i} - E_{i,i+1}\right) \\ & J_i^{(n)} & 0_n \\ & & J_i^{(n)} \end{pmatrix} \text{ and } Y = \begin{pmatrix} J_i^{(n)} & a\left(E_{i+1,i} - E_{i,i+1}\right) & a^2E_{i+1,i} \\ & J_i^{(n)} & a\left(E_{i+1,i} - E_{i,i+1}\right) \\ & & J_i^{(n)} \end{pmatrix}$$

are involutions in $\mathcal{C}(S)$.

Proof. By Corollary 2.6, X and Y belong to $\mathcal{C}(S)$. Next, we set

$$J_1 = a\left[J_i^{(n)}\left(E_{i+1,i} - E_{i,i+1}\right) + \left(E_{i+1,i} - E_{i,i+1}\right)J_i^{(n)}\right],$$

$$J_2 = a^2\left[J_i^{(n)}E_{i+1,i} + \left(E_{i+1,i} - E_{i,i+1}\right)^2 + E_{i+1,i}J_i^{(n)}\right].$$

By Proposition 3.1, it follows that $J_1 = J_2 = 0_n$. Thus,

$$X^2 = \begin{pmatrix} \left(J_i^{(n)}\right)^2 & 0_n & J_1 \\ & \left(J_i^{(n)}\right)^2 & 0_n \\ & & \left(J_i^{(n)}\right)^2 \end{pmatrix} = \begin{pmatrix} \mathrm{I}_n & 0_n & 0_n \\ & \mathrm{I}_n & 0_n \\ & & \mathrm{I}_n \end{pmatrix} = \mathrm{I}_{3n},\ Y^2 = \begin{pmatrix} \left(J_i^{(n)}\right)^2 & J_1 & J_2 \\ & \left(J_i^{(n)}\right)^2 & J_1 \\ & & \left(J_i^{(n)}\right)^2 \end{pmatrix} = \begin{pmatrix} \mathrm{I}_n & 0_n & 0_n \\ & \mathrm{I}_n & 0_n \\ & & \mathrm{I}_n \end{pmatrix} = \mathrm{I}_{3n}.$$

Therefore, X and Y are involutions in $\mathcal{C}(S)$.

Lemma 3.5. *Let $S \in \mathrm{M}_{3n}(\mathbb{F})$ be a Weyr matrix with a homogeneous Weyr structure of length 3 for $n \geq 2$. Then, for any $a \in \mathbb{F}$ and any $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$, the matrices*

$$X = \begin{pmatrix} \mathrm{I}_n & 0_n & aE_{i,j} \\ & \mathrm{I}_n & 0_n \\ & & \mathrm{I}_n \end{pmatrix} \text{ and } Y = \begin{pmatrix} \mathrm{I}_n & aE_{i,j} & 0_n \\ & \mathrm{I}_n & aE_{i,j} \\ & & \mathrm{I}_n \end{pmatrix}$$

are commutators of involutions in $\mathcal{C}(S)$.

Proof. Direct computation shows that $X = [X_1, X_2]$ is a commutator of two matrices X_1 and X_2, where $X_1 = \begin{pmatrix} D_i^{(n)} & 0_n & \frac{a}{2}E_{i,j} \\ & D_i^{(n)} & 0_n \\ & & D_i^{(n)} \end{pmatrix}$ and $X_2 = \begin{pmatrix} D_i^{(n)} & 0_n & 0_n \\ & D_i^{(n)} & 0_n \\ & & D_i^{(n)} \end{pmatrix}$. Moreover, by Proposition

3.3, both X_1 and X_2 are involutions in $\mathcal{C}(S)$. Therefore, X is a commutator of involutions in $\mathcal{C}(S)$. Similarly, $Y=[Y_1,Y_2]$ is also a commutator of two involutions Y_1 and Y_2 in $\mathcal{C}(S)$, where $Y_1=\begin{pmatrix} D_i^{(n)} & \frac{a}{2}E_{i,j} & 0_n \\ & D_i^{(n)} & \frac{a}{2}E_{i,j} \\ & & D_i^{(n)} \end{pmatrix}$ and $Y_2=\begin{pmatrix} D_i^{(n)} & 0_n & 0_n \\ & D_i^{(n)} & 0_n \\ & & D_i^{(n)} \end{pmatrix}$.

Lemma 3.6. *Let $S\in \mathrm{M}_{3n}(\mathbb{F})$ be a Weyr matrix with a homogeneous Weyr structure of length 3 for $n\geq 2$. Then, for any $a\in\mathbb{F}$ and any $i\in\{1,2,\ldots,n-1\}$, the matrix*

$$X=\begin{pmatrix} \mathrm{I}_n & 0_n & a\left(E_{i+1,i+1}-E_{i,i}\right) \\ & \mathrm{I}_n & 0_n \\ & & \mathrm{I}_n \end{pmatrix}$$

is a commutator of involutions in $\mathcal{C}(S)$.

Proof. Direct computation shows that the matrix X is a commutator of two matrices X_1 and X_2, where

$$X_1=\begin{pmatrix} J_i^{(n)} & 0_n & \frac{a}{2}\left(E_{i+1,i}-E_{i,i+1}\right) \\ & J_i^{(n)} & 0_n \\ & & J_i^{(n)} \end{pmatrix},\ X_2=\begin{pmatrix} J_i^{(n)} & 0_n & 0_n \\ & J_i^{(n)} & 0_n \\ & & J_i^{(n)} \end{pmatrix}.$$

By Proposition 3.4, both X_1 and X_2 are involutions in $\mathcal{C}(S)$. Thus, X can be expressed as a commutator of involutions in $\mathcal{C}(S)$.

Lemma 3.7. *Let $S\in \mathrm{M}_{3n}(\mathbb{F})$ be a Weyr matrix with a homogeneous Weyr structure of length 3 for $n\geq 2$, and $A\in \mathrm{M}_n(\mathbb{F})$ be a matrix with trace zero. Then the matrix*

$$X=\begin{pmatrix} \mathrm{I}_n & 0_n & A \\ & \mathrm{I}_n & 0_n \\ & & \mathrm{I}_n \end{pmatrix}$$

can be expressed as a product of at most n^2-1 commutators of involutions in $\mathcal{C}(S)$.

Proof. Suppose $A=\left(a_{ij}\right)\in \mathrm{M}_n(\mathbb{F})$ with $\mathrm{tr}(A)=0$, that is, $\sum_{i=1}^{n}a_{ii}=0$. For each positive integer $s\in\{1,2,\ldots,n\}$, we set $a_s=\sum_{i=1}^{s}a_{ii}$. For each $i\in\{1,2,\ldots,n-1\}$, we consider the matrix $X_i=\begin{pmatrix} \mathrm{I}_n & 0_n & a_i\left(E_{i+1,i+1}-E_{i,i}\right) \\ & \mathrm{I}_n & 0_n \\ & & \mathrm{I}_n \end{pmatrix}$. Let $X=\begin{pmatrix} \mathrm{I}_n & 0_n & A \\ & \mathrm{I}_n & 0_n \\ & & \mathrm{I}_n \end{pmatrix}$ and $Y=X_{n-1}X_{n-2}\ldots X_2X_1X$. It follows that

$$Y = \begin{pmatrix} \mathrm{I}_n & 0_n & Y' \\ & \mathrm{I}_n & 0_n \\ & & \mathrm{I}_n \end{pmatrix},$$

where $Y' = A + \sum_{i=1}^{n-1} a_i \left(E_{i+1,i+1} - E_{i,i} \right) \in \mathrm{M}_n(\mathbb{F})$. Combining Proposition 3.2 with $\mathrm{tr}(A) = 0$ gives $Y' = \sum_{1 \le i \ne j \le n} a_{ij} E_{i,j}$. It follows that Y can be written as

$$Y = \prod_{1 \le i \ne j \le n} \begin{pmatrix} \mathrm{I}_n & 0_n & a_{ij} E_{i,j} \\ & \mathrm{I}_n & 0_n \\ & & \mathrm{I}_n \end{pmatrix}.$$

According to Lemma 3.5, Y is a product of at most $n^2 - n$ commutators of involutions in $\mathcal{C}(S)$. By Lemma 3.6, X_i is a commutator of involutions in $\mathcal{C}(S)$ for all $i \in \{1, 2, \ldots, n-1\}$. Therefore, applying Remark 2.2, it follows that X_i^{-1} is also a commutator of involutions in $\mathcal{C}(S)$ for all $i \in \{1, 2, \ldots, n-1\}$. Thus, $X = X_1^{-1} X_2^{-1} \ldots X_{n-1}^{-1} Y$ is the product of at most $n^2 - 1$ commutators of involutions in $\mathcal{C}(S)$.

To establish the next lemma, we use the following theorem from De La Cruz and Tañedo (2022).

Theorem 3.8. (De La Cruz & Tañedo, 2022, Theorem 3.3) Let $S \in \mathrm{M}_{3n}(\mathbb{F})$ be a Weyr matrix with a homogeneous Weyr structure of length 3 for $n \ge 2$. Suppose $X_i \in \mathcal{C}(S)$ for all $i = 1, 2, \ldots, s$ satisfies the following properties:

(P1): $\det X_{1,1} = \pm 1$;

(P2): $\mathrm{tr}\left(X_{1,1}^{-1} X_{1,2} \right) = 0$; and

(P3): $2\mathrm{tr}\left(X_{1,1}^{-1} X_{1,3} \right) = \mathrm{tr}\left(\left(X_{1,1}^{-1} X_{1,2} \right)^2 \right)$.

Then $X = \prod_{1}^{s} X_i$ also satisfies (P1), (P2), and (P3).

Lemma 3.9. *Let $S \in \mathrm{M}_{3n}(\mathbb{F})$ be a Weyr matrix with a homogeneous Weyr structure of length 3 for $n \ge 2$. Suppose that*

$$X = \begin{pmatrix} \mathrm{I}_n & A & B \\ & \mathrm{I}_n & A \\ & & \mathrm{I}_n \end{pmatrix} \in \mathcal{C}(S),$$

where $A, B \in \mathrm{M}_n(\mathbb{F})$ such that $\mathrm{tr}(A) = 0$ and $2\mathrm{tr}(B) = \mathrm{tr}(A^2)$. Then, the matrix X can be expressed as a product of at most $2n^2$ commutators of involutions in $\mathcal{C}(S)$.

Proof. Suppose X is the matrix defined in the statement of Lemma 3.9 and $A=\left(a_{ij}\right)\in \mathrm{M}_n\left(\mathbb{F}\right)$. For each $s\in\left\{1,2,\ldots,n\right\}$, set $a_s=\sum_{i=1}^{s}a_{ii}$. For each $i\in\left\{1,2,\ldots,n-1\right\}$, we define $X_i=\begin{pmatrix} \mathrm{I}_n & a_i\left(E_{i+1,i+1}-E_{i,i}\right) & a_i^2E_{i+1,i+1} \\ & \mathrm{I}_n & a_i\left(E_{i+1,i+1}-E_{i,i}\right) \\ & & \mathrm{I}_n \end{pmatrix}$. Direct computation shows that $X_i=\left[Y_i,Z_i\right]$, where Y_i and Z_i are given by

$$Y_i=\begin{pmatrix} J_i^{(n)} & -\frac{3a_i}{4}\left(E_{i+1,i}-E_{i,i+1}\right) & \frac{9a_i^2}{16}E_{i+1,i} \\ & J_i^{(n)} & -\frac{3a_i}{4}\left(E_{i+1,i}-E_{i,i+1}\right) \\ & & J_i^{(n)} \end{pmatrix},\ Z_i=\begin{pmatrix} J_i^{(n)} & -\frac{a_i}{4}\left(E_{i+1,i}-E_{i,i+1}\right) & \frac{a_i^2}{16}E_{i+1,i} \\ & J_i^{(n)} & -\frac{a_i}{4}\left(E_{i+1,i}-E_{i,i+1}\right) \\ & & J_i^{(n)} \end{pmatrix}.$$

Moreover, by Proposition 3.4, Y_i and Z_i are involutions in $\mathcal{C}\left(S\right)$ for all $i\in\left\{1,2,\ldots,n-1\right\}$, implying that X_i is a commutator of involutions in $\mathcal{C}\left(S\right)$. Additionally, by Remark 2.2, X_i^{-1} is also a commutator of involutions in $\mathcal{C}\left(S\right)$ for all $i\in\left\{1,2,\ldots,n-1\right\}$. For each $i,j\in\left\{1,2,\ldots,n\right\}$ and $i\neq j$, we set

$$Y_{ij}=\begin{pmatrix} \mathrm{I}_n & -a_{ij}E_{ij} & 0_n \\ & \mathrm{I}_n & -a_{ij}E_{ij} \\ & & \mathrm{I}_n \end{pmatrix}.$$

Then, we have

$$Z=X_{n-1}X_{n-2}\ldots X_2X_1X\left(\prod_{i\neq j}Y_{ij}\right)=\begin{pmatrix} \mathrm{I}_n & A_1 & B_1 \\ & \mathrm{I}_n & A_1 \\ & & \mathrm{I}_n \end{pmatrix},$$

where $A_1=A+\sum_{i=1}^{n-1}a_i\left(E_{i+1,i+1}-E_{i,i}\right)-\sum_{i\neq j}a_{ij}E_{ij}$ and $B_1\in\mathrm{M}_n\left(\mathbb{F}\right)$. By Proposition 3.2, $A_1=0_n$. Applying Theorem 3.10 gives $2\mathrm{tr}\left(B_1\right)=\mathrm{tr}\left(A_1^2\right)=0$, hence $\mathrm{tr}\left(B_1\right)=0$. By Lemma 3.7, Z is a product of at most n^2-1 commutators of involutions in $\mathcal{C}\left(S\right)$. Furthermore, by Lemma 3.5, Y_{ij} is a commutator of involutions in $\mathcal{C}\left(S\right)$ for all $i\neq j$. This implies that $\prod_{i\neq j}Y_{ij}$ is a product of n^2-n commutators of involutions in $\mathcal{C}\left(S\right)$. Furthermore, by Remark 2.2, $\left(\prod_{i\neq j}Y_{ij}\right)^{-1}$ is also a product of n^2-n commutators of involutions in $\mathcal{C}\left(S\right)$. Consequently, $X=X_1^{-1}X_2^{-1}\ldots X_{n-2}^{-1}X_{n-1}^{-1}Z\left(\prod_{i\neq j}Y_{ij}\right)^{-1}$ can be expressed as a product of at most $2n^2-2$ commutators of involutions in $\mathcal{C}\left(S\right)$.

Lemma 3.10. *Let* $S \in \mathrm{M}_{3n}(\mathbb{F})$ *be a Weyr matrix with a homogeneous Weyr structure of length 3 and* $n \geq 2$. *If the matrix* $X \in \mathcal{I}_{\mathcal{C}(S)}$, *then the matrix* X *has the form*

$$X = \begin{pmatrix} X_1 & X_2 & X_3 \\ & X_1 & X_2 \\ & & X_1 \end{pmatrix},$$

where $X_1 \in \mathrm{SL}_n(\mathbb{F}), X_2, X_3 \in \mathrm{M}_n(\mathbb{F})$, *such that* $\mathrm{tr}\left(X_1^{-1}X_2\right) = 0$ *and* $2\mathrm{tr}\left(X_1^{-1}X_3\right) = \mathrm{tr}\left(\left(X_1^{-1}X_2\right)^2\right)$.

Proof. Assume $X \in \mathcal{I}_{\mathcal{C}(S)}$, that is, X can be expressed as a product of commutators of involutions in $\mathcal{C}(S)$. Since each commutator of involutions in $\mathcal{C}(S)$ is a product of two involutions in $\mathcal{C}(S)$, it follows that X is also a product of involutions in $\mathcal{C}(S)$. According to Theorem 1.3, we have

$$X = \begin{pmatrix} X_1 & X_2 & X_3 \\ & X_1 & X_2 \\ & & X_1 \end{pmatrix},$$

where $\det X_1 = \pm 1$, $\mathrm{tr}\left(X_1^{-1}X_2\right) = 0$, and $2\mathrm{tr}\left(X_1^{-1}X_3\right) = \mathrm{tr}\left(\left(X_1^{-1}X_2\right)^2\right)$. In addition, since $\det X = 1$, it follows that $\det X_1 = 1$. Thus, $X_1 \in \mathrm{SL}_n(\mathbb{F})$, which completes the proof.

We conclude by proving Theorem 1.6.

Proof of Theorem 1.6.

Case 1: Let $S \in \mathrm{M}_3(\mathbb{F})$ be a Weyr matrix with a homogeneous Weyr structure of length 3. By Corollary 2.6, a matrix $X \in \mathrm{M}_3(\mathbb{F})$ belongs to $\mathcal{C}(S)$ if and only if X has the form $X = \begin{pmatrix} a & b & c \\ & a & b \\ & & a \end{pmatrix}$, where $a, b, c \in \mathbb{F}$. Moreover, X is an involution in $\mathcal{C}(S)$ if and only if $X^2 = \mathrm{I}_3$, that is, $\begin{pmatrix} a^2 & 2ab & b^2+2ac \\ & a^2 & 2ab \\ & & a^2 \end{pmatrix} = \mathrm{I}_3$. Thus, $a = \pm 1$ and $b = c = 0$, hence $X = \pm \mathrm{I}_3$. In this case, $\mathcal{I}_{\mathcal{C}(S)} = \{\mathrm{I}_3\}$, so Theorem 1.6 holds when S is a Weyr matrix with the Weyr structure $(1,1,1)$.

Case 2: Let $S \in \mathrm{M}_{3n}(\mathbb{F})$ be a Weyr matrix with a homogeneous Weyr structure of length 3, with $n \geq 2$. Assume $X = \begin{pmatrix} X_1 & X_2 & X_3 \\ & X_1 & X_2 \\ & & X_1 \end{pmatrix}$, where $X_1 \in \mathrm{SL}_n(\mathbb{F})$, $\mathrm{tr}\left(X_1^{-1}X_2\right) = 0$, and $2\mathrm{tr}\left(X_1^{-1}X_3\right) = \mathrm{tr}\left(\left(X_1^{-1}X_2\right)^2\right)$. We have

$$X = \begin{pmatrix} X_1 & 0_n & 0_n \\ & X_1 & 0_n \\ & & X_1 \end{pmatrix} \begin{pmatrix} \mathrm{I}_n & X_1^{-1}X_2 & X_1^{-1}X_3 \\ & \mathrm{I}_n & X_1^{-1}X_2 \\ & & \mathrm{I}_n \end{pmatrix}.$$

Since $X_1 \in \mathrm{SL}_n(\mathbb{F})$, Theorem 1.1 implies that X_1 is a product of at most two commutators of involutions in $\mathrm{GL}_n(\mathbb{F})$. Therefore, we can write X_1 as $X_1 = [X_{1,1}, X_{1,2}][X_{2,1}, X_{2,2}]$, where $X_{i,j} \in \mathrm{GL}_n(\mathbb{F})$, and $X_{i,j}^2 = \mathrm{I}_n$ for all $i, j \in \{1,2\}$. For each $i, j \in \{1,2\}$, we set $Y_{i,j} = \begin{pmatrix} X_{i,j} & 0_n & 0_n \\ & X_{i,j} & 0_n \\ & & X_{i,j} \end{pmatrix}$. By Corollary 2.6, $Y_{i,j} \in \mathcal{C}(S)$ for all $i, j \in \{1,2\}$. Moreover, we have

$$(Y_{i,j})^2 = \begin{pmatrix} (X_{i,j})^2 & 0_n & 0_n \\ & (X_{i,j})^2 & 0_n \\ & & (X_{i,j})^2 \end{pmatrix} = \begin{pmatrix} \mathrm{I}_n & 0_n & 0_n \\ & \mathrm{I}_n & 0_n \\ & & \mathrm{I}_n \end{pmatrix} = \mathrm{I}_{3n}.$$

It follows that $Y_{i,j}$ is an involution in $\mathcal{C}(S)$ for all $i, j \in \{1,2\}$. Therefore,

$$\begin{pmatrix} X_1 & 0_n & 0_n \\ & X_1 & 0_n \\ & & X_1 \end{pmatrix} = [Y_{1,1}, Y_{1,2}][Y_{2,1}, Y_{2,2}]$$

is a product of at most two commutators of involutions in $\mathcal{C}(S)$. In addition, since $\mathrm{tr}(X_1^{-1}X_2) = 0$ and $2\mathrm{tr}(X_1^{-1}X_3) = \mathrm{tr}\left((X_1^{-1}X_2)^2\right)$, according to Lemma 3.8, we get

$$\begin{pmatrix} \mathrm{I}_n & X_1^{-1}X_2 & X_1^{-1}X_3 \\ & \mathrm{I}_n & X_1^{-1}X_2 \\ & & \mathrm{I}_n \end{pmatrix}$$

is a product of at most $2n^2 - 2$ commutators of involutions in $\mathcal{C}(S)$. As a result, X can be expressed as a product of at most $2n^2$ commutators of involutions in $\mathcal{C}(S)$.

4. Conclusion

We have established a necessary and sufficient condition for a matrix over an algebraically closed field of characteristic different from 2 to be expressed as a product of commutators of involutions in the centralizer of a basic Weyr matrix with a homogeneous Weyr structure of length 3. The case of greater length is considerably more challenging and will be the subject of future work.

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VỀ SỰ PHÂN TÍCH CÁC MA TRẬN TRONG TÂM HOÁ TỬ CỦA MỘT MA TRẬN WEYR

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TÓM TẮT

Cho F *là một trường đóng đại số có đặc số khác* 2, *và* n *là một số nguyên dương. Giả sử* $S \in \mathrm{M}_{3n}(\mathrm{F})$ *là một ma trận Weyr cơ sở có cấu trúc Weyr thuần nhất với độ dài* 3, *và* $C(S)$ *là tâm hóa tử của* S *trong* $\mathrm{M}_{3n}(\mathrm{F})$. *Bài báo này tập trung mô tả nhóm con được sinh bởi các hoán tử của các ma trận đối hợp trong* $C(S)$. *Hơn nữa, chúng tôi chứng minh rằng mọi ma trận trong nhóm con này đều có thể biểu diễn được thành tích của nhiều nhất* $2n^2$ *hoán tử của các ma trận đối hợp trong* $C(S)$.

Từ khoá: hoán tử; đối hợp; tâm hoá tử ma trận; sự phân tích ma trận; ma trận Weyr