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Research Article

HARDY INEQUALITIES WITH HI-POTENTIAL INVOLVED DUNKL OPERATOR

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ABSTRACT

We prove a Hardy-type inequalities in Dunkl setting, integrated with an HI-potential. Our approach utilizes the h-harmonic expansion of functions f ä $L^2(m_k)$ *and integrating techniques such as integral transformations, spherical coordinate formulas, and separation of variables, we derive the main result presented in Theorem 1. These outcomes build upon and extend the foundational work of Ghoussoub and Moradifam (2013), which addressed Hardy-type inequalities involving the Laplace operator and the Lebesgue measure in conjunction with an HI-potential. Consequently, our findings advance the generalization of Hardy inequality within broader context of Dunkl theory. Moreover, this research carries substantial implications for analyzing differential equations and partial differential equations that exhibit singularities, thereby providing enhanced understanding of the qualitative properties and behaviors of solutions in these equation classes. This extension not only refines existing inequalities but also opens avenues for applications in mathematical physics and functional analysis.*

Keywords: best constant; Hardy inequality; HI-potential

1. Introduction

Functional inequalities play a crucial role in mathematical fields such as partial differential equations functional analysis, and mathematical physics, providing critical insights into the behavior of solutions near singularities. Many mathematicians have explored these inequalities from diverse perspectives. The primary focus of this article is the renowned Hardy inequality, which holds significant importance across various domains, including mathematical analysis, probability theory, and partial differential equations.

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$$\int_{\mathbb{R}^N} |\nabla f|^2 dx \geq \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|f|^2}{|x|^2} dx.$$

A particular equality, as derived in Barbatis et al. (2004) and Dolbeault and Volzone (2012), delineates the configuration of the remainder terms that vanish, thus affording a precise elucidation of Hardy inequalities and the absence of non-trivial maximizers:

$$\int_{\mathbb{R}^N} |\nabla f|^2 dx = \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|f|^2}{|x|^2} dx + \int_{\mathbb{R}^N} \left|\nabla f + \left(\frac{N-2}{2}\right)\frac{f}{|x|}\frac{x}{|x|}\right|^2 dx.$$

Note that the additional term $\int_{\mathbb{R}^N} \left|\nabla f + \left(\frac{N-2}{2}\right)\frac{f}{|x|}\frac{x}{|x|}\right|^2 dx = 0$ when $f(z) = a|z|^{-\frac{N-2}{2}}$.

Meanwhile the integral $\int_{\mathbb{R}^N} \frac{|f|^2}{|x|^2} dx$ is diverges unless $a = 0$. Consequently, the Hardy inequality admits a virtual optimizer of the form $|x|^{-\frac{N-2}{2}}$.

Note that, the sharp constant $\left(\frac{N-2}{2}\right)^2$ unattainable by any non-zero function in the space, necessitating enhancements via the inclusion of non-negative component on the right-hand expression. On the whole space $\mathbb{R}^N$, Ghoussoub et al (2011) demonstrated that no strictly positive function $A \in C^1(0,\infty)$ such that

$$\int_{\mathbb{R}^N} |\nabla f|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|f|^2}{|x|^2} dx \geq \int_{\mathbb{R}^N} A(|x|)|f|^2 dx, \ \forall f \in C_0^\infty(\mathbb{R}^N).$$

Nevertheless, on bounded domains, supplementary terms can be included. For example, let $\Omega \subset \mathbb{R}^N, N \geq 3$ with $0 \in \Omega$ and Ω bounded domain in $\mathbb{R}^N$. Brezis and Vázquez (1997) established that

$$\int_{\Omega} |\nabla f|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_{\Omega} \frac{|f|^2}{|x|^2} dx \geq z_0^2 V_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}} \int_{\Omega} |f|^2 dx, \forall u \in W_0^{1,2}(\Omega):$$

where $V_N = mes(B)$ and $z_0 = 2.4048...$ is the first term of the Bessel function $K_0(z)$. In case Ω is a ball, we have $z_0^2 V_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}}$ is sharp.

The constants $\left(\frac{N-2}{2}\right)^2$ and $\frac{z_0^2}{R^2}$ are sharp with equality achieved only in the trivial case $u = 0$, though they correspond to respective virtual optimizers $\frac{K_{0;R}(|x|)}{|x|^{\frac{N-2}{2}}} f\left(\frac{x}{|x|}\right)$ and $\frac{K_{0;R}(|x|)}{|x|^{\frac{N-2}{2}}}$

respectively. Since the optimal constant $z_0^2 V_N^{\frac{2}{N}} |\Omega|^{-\frac{2}{N}}$ is unattained on the whole space $W_0^{1,2}(\Omega)$, it is plausible to propose that the first term functions simply as the starting point of an endless series of supplementary terms that could be added to the right-hand expression of the Hardy inequality. This topic has been examined by various scholars, featuring standard references such as Adimurthi et al. (2002), Barbatis et al. (2003, 2004), Cazacu and Zuazua (2013), Davies (1999), Evans and Lewis (2007), and Ghoussoub and Moradifam (2011, 2013). Notably, Filippas and Tertikas (2002) introduced an infinite series expansion for the Hardy inequality, deemed optimal in a specific sense. Additionally, it is noteworthy that, aiming to enhance, broaden, and consolidate various outcomes in this area, Ghoussoub et al (2013) proposed the concept HI-potentials (HIP) and examined their associations with Hardy inequalities. A prominent result from their work is presented below:

Theorem A. (Ghoussoub et al., 2013) *Let $K \in C^1(0,R)$ be a nonnegative function that is monotonically decreasing.. The ensuing statements hold interchangeably:*

(1) K is HIP on $(0,R)$, that is $\exists s(r) > 0 : s''(r) + \frac{1}{r}s'(r) + K(r)s(r) = 0$ has a solution on $(0,R)$.

(2) $$\int_{B_R} |\nabla f|^2 dx - \left(\frac{N-2}{2}\right)^2 \int_{B_R} \frac{|f|^2}{|x|^2} dx \geq \int_{B_R} K(|x|)|f|^2 dx, \forall u \in W_0^{1,2}(B_R).$$

Here are a few immediate examples of such functions:

- $K \equiv 0$ is a HIP on $(0,R)$. Indeed, $z''(r) + \frac{1}{r}z'(r) = 0$ has positive solution $z(r) = -\ln(\frac{e}{R}r)$ on $(0,R)$

- $K \equiv 1$ is a HIP on $(0,z_0)$ where $z_0 = 2.4048......$ is the first term of the Bessel function $K_0(z)$. Indeed, the latter is a positive solution of $s''(r) + \frac{1}{r}s'(r) + s(r) = 0$ until it reaches its first zero at z $_0$

Dunkl setting

Dunkl theory, essential tools in generalized harmonic analysis and representation theory, find broad applications ranging from theoretical physics to probability and statistics.

We will now briefly introduce Dunkl's theory. For more details on Dunkl's theory, the interested reader is invited to consult, for example, Rösler (2003).

For a vector $b \in \mathbb{R}^N \setminus \{0\}$, The reflection operation s_b is specified relative to the hyperplane perpendicular to a nonzero vector $\langle b \rangle^{\perp}$:

$$\sigma_\beta(y) = y - 2\langle \beta, y\rangle \frac{\beta}{|\beta|^2},\ y \in \mathbb{R}^N.$$

Consider a finite collection $R \subset \mathbb{R}^N \setminus \{0\}$ satisfying $R \cap \mathbb{R}_\alpha = \{\pm\beta\}$ for every $\beta \in R$; such a set constitutes a root system within the framework of Dunkl setting.

A multiplicity function $h: R \to \mathbb{C}$ is termed B-invariant if $h(\sigma_\alpha(\beta)) = h(\beta)$ whenever $\sigma_\beta \in G$ acts on $\beta \in R$. The root system R is partitioned as $R = R_+ \cup (-R_+)$ wherein the subset R_+ and $-R_+$ are demarcated by a flat surface cutting through the center point. It is apparent that this partitioning lacks uniqueness. Nonetheless, owing to the invariance of the multiplicity function k under the group B, the precise election of R_+ exerts no influence on the ensuing definition of the Dunkl weight.

In this article, we consider the Dunkl weight $\mu_k(x)$, we have $\mu_k(x)$ be a B-invariant with degree $2\gamma_k = \sum_{\beta \in R_+} 2k(\beta)$:

$$\mu_k(x) = \prod_{\beta \in R_+} |\langle \beta, x\rangle|^{2k(\beta)}.$$

This weight $\mu_k(x)$ has a important role in Dunkl setting. Hereinafter, the assumption $k(\alpha) \geq 0$ is maintained and $d_k = N + 2\gamma_k$ and denote $\mu_k(x)dx = d\mu_k(x)$

For $j \in \{1,2,\ldots,N\}$ The Dunkl operators is defined by

$$T_j f(z) = \frac{\partial f}{\partial z_j} + \sum_{\beta \in R_+} k(\beta)\beta_j \frac{f(z) - f(\sigma_\beta z)}{\langle \beta, z\rangle}$$

with $\beta = (\beta_1, \beta_2, .., \beta_N)$. These Dunkl operators extend the classical partial derivatives in standard analysis.

Analogously, the Dunkl gradient is formulated as $\nabla_k = (T_1, T_2, .., T_N)$ while the Dunkl Laplacian is given by $\Delta_k = \sum_{j=1}^{k} T_j^2$.

For any $h, k \in C^1(\mathbb{R}^N)$ at least one of f or g is G-invariant and for every $1 \leq j \leq N$, we have $T_j(hk) = hT_j k + kT_j h$.

Regarding the Dunkl weight, an integration-by-parts formula is available, facilitating the handling of boundary terms and symmetry properties in associated integrals

$$\int_{\mathbb{R}^N} T_j(h)k\,d\mu_k(x) = -\int_{\mathbb{R}^N} T_j(k)h\,d\mu_k(x)$$

Spherical k-harmonic

Next, a concise overview of k-harmonics is provided; this discussion draws from the work of Rösler (2003), and readers seeking further elaboration are encouraged to consult the original source for comprehensive details. An k-harmonic polynomial is a degree-k homogeneous polynomial associated with the h-harmonic operator $\Delta_k p = 0$

Spherical k-harmonic with m degree are defined to be restrictions of k-harmonic polynomials of degree m to the sphere S^{N-1}. Let P_n the space of k-harmonic of degree. Denote $\dim(m)$ the dimension of P_n. Moreover, with n = 1,2,3,… the space $L^2(S^{N-1}, d\mu_k(x)) = \oplus P_n$, let Z_i^n, $i = 1,..,\dim(n)$ serve as an orthonormal basis for the space P_n. With $x = r\xi$ for r $\in (0,+\infty)$ and $\xi \in S^{N-1}$, we have

$$\Delta_k = \frac{\partial^2}{\partial r^2} + \frac{N+2\gamma-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_{k,0}$$

Where Δ_k is Dunkl Laplacian and $\Delta_{k,0}$ serves as a form of Laplace-Beltrami operator on S^{N-1}. We have

$$\Delta_{k,0} Y = -l(l+N+2\gamma-2)Y = \lambda_l Y$$

Let $f \in L^2(\mu_k(x))$ we have the k-harmonic expansion of function *f*:

$$f(r\xi) = \sum_{n=0}^{\infty}\sum_{i=1}^{d(n)} f_{n,i}(r) Y_i^n(\xi)$$

where

$$fu_{n,i}(r) = \int_{S^{N-1}} f(r\xi) Y_i^n(\xi) h_k(\xi) d\sigma(\xi)$$

and σ is the measure on S^{N-1}

2. Main results

The objective of this study is to develop an enhanced formulation of the preceding outcome within the Dunkl theoretical context.

Theorem 1. *Let $N_\gamma > 2$ Suppose K is a normalized HIP on $(0,R)$, meaning that $z''(r) + \frac{1}{r}z'(r) + K(r)z(r) = 0$ possesses a solution $z(r) > 0$ on (0,R), the following inequality applies to every function f $\in C_c^\infty(\Omega)$*

$$\int_\Omega \left|\nabla_k f\right|^2 d\mu_k(x) - \left(\frac{N_\gamma - 2}{2}\right)^2 \int_\Omega \frac{|f|^2}{|x|^2} d\mu_k(x) \geq \int_\Omega P(|x|) f^2(x) d\mu_k(x).$$

where $N_\gamma = N + 2\gamma$.

Proof

Let $f \in L^2(\mu_k)$ we have

$$f(x)=f(r\xi)=\sum_{n=0}^{\infty}\sum_{i=1}^{d(n)}u_{n,i}(r)Y_i^n(\xi),$$

where

$$f_{n,i}(r)=\int_{S^{N-1}}f(r\xi)Y_i^n(\xi)\omega_k(\xi)d\sigma(\xi)$$

Note that

$$\begin{aligned}\int_{\Omega}\left|\nabla_k f\right|^2 d\mu_k(x)&=-\int_{\Omega}f\Delta_k f d\mu_k(x)\\&=-\int_{\Omega}f\left(\frac{\partial^2 f}{\partial r^2}+\frac{N+2\gamma-1}{r}\frac{\partial f}{\partial r}+\frac{1}{r^2}\Delta_{k,0}f\right)d\mu_k(x)\end{aligned}$$

Using the orthogonality of the h-harmonic $\{Y_i^n\}$

$$\begin{aligned}&\int_{\Omega}\left|\nabla_k f\right|^2 d\mu_k(x)\\&=-\sum_{n=0}^{\infty}\sum_{i=1}^{d(n)}\int_{\mathbb{R}^n}\left(f_{n,i}(r)f''_{n,i}(r)+\frac{N+2\gamma-1}{r}f'_{n,i}(r)f_{n,i}(r)+\frac{\lambda_n}{r^2}f_{n,i}^2(r)\right)d\mu_k(x)\\&=-\sum_{n=0}^{\infty}\sum_{i=1}^{d(n)}\int_{S^{N-1}}\int_0^R\left(f_{n,i}(r)u''_{n,i}(r)+\frac{N+2\gamma-1}{r}f'_{n,i}(r)u_{n,i}(r)+\frac{\lambda_n}{r^2}f_{n,i}^2(r)\right)r^{N+2\gamma-1}drd\mu_k(x)\\&=-\left|S^{N-1}\right|\sum_{n=0}^{\infty}\sum_{i=1}^{d(n)}\int_0^R\left(f_{n,i}(r)f''_{n,i}(r)+\frac{N+2\gamma-1}{r}f'_{n,i}(r)f_{n,i}(r)+\frac{\lambda_n}{r^2}f_{n,i}^2(r)\right)r^{N+2\gamma-1}dr\\&=\sum_{n=0}^{\infty}\sum_{i=1}^{d(n)}\int_{\mathbb{R}^n}\left(\left|\nabla_k f_{n,i}\right|^2-\lambda_n\frac{f_{n,i}^2}{|x|^2}\right)d\mu_k(x)\end{aligned}$$

Let $\Lambda=\left(\frac{N_\gamma-2}{2}\right)^2$. Using the fact that $\lambda_n\le 0,\forall n,$ we have

$$\begin{aligned}\int_{\Omega}\left[\left|\nabla_k f\right|^2-\Lambda\frac{f^2}{|x|^2}\right]d\mu_k(x)&=\sum_{n=0}^{\infty}\sum_{i=1}^{d(n)}\int_{\Omega}\left(\left|\nabla_k f_{n,i}\right|^2-\left(\lambda_n+\Lambda\right)\frac{f_{n,i}^2}{|x|^2}\right)d\mu_k(x)\\&=\left|S^{N-1}\right|\sum_{n=0}^{\infty}\sum_{i=1}^{d(n)}\int_0^R\left(\left|\nabla_k f_{n,i}\right|^2-\left(\lambda_n+\Lambda\right)\frac{f_{n,i}^2}{|x|^2}\right)r^{N_\gamma-1}dr\\&\ge\left|S^{N-1}\right|\sum_{n=0}^{\infty}\sum_{i=1}^{d(n)}\int_0^R\left(\left|\nabla_k f_{n,i}\right|^2-\Lambda\frac{f_{n,i}^2}{|x|^2}\right)r^{N_\gamma-1}dr\end{aligned}$$

Thus

$$\int_{\Omega}\left[\left|\nabla_k f\right|^2-\Lambda\frac{f^2}{|x|^2}\right]d\mu_k(x)\ge\left|S^{N-1}\right|\sum_{n=0}^{\infty}\sum_{i=1}^{d(n)}\int_0^R\left(\left|\nabla_k f_{n,i}\right|^2-\left(\frac{N_\gamma-2}{2}\right)^2\frac{f_{n,i}^2}{|x|^2}\right)r^{N_\gamma-1}dr.$$

We estimate $\int_0^R \left(\left|\nabla_k f_{n,i}\right|^2 - \left(\frac{N_g-2}{2}\right)^2 \frac{f_{n,i}^2}{|x|^2}\right) r^{N_g-1}dr$ as follows.

Define $g(r) = f_{n,i}(r) r^{\frac{N_g-2}{2}}$ we have

$$\int_0^R \left(\left|\nabla_k f_{n,i}\right|^2 - \left(\frac{N_g-2}{2}\right)^2 \frac{f_{n,i}^2}{r^2}\right) r^{N_g-1}dr$$

$$= \int_0^R \left(\frac{N_g-2}{2} r^{-\frac{N_g}{2}} g(r) - r^{1-\frac{N_g}{2}} g'(r)\right)^2 r^{N_g-1}dr - \left(\frac{N_g-2}{2}\right)^2 \int_0^R \frac{g^2}{r^{N_g}} r^{N_g-1}dr$$

$$= \left(\frac{N_g-2}{2}\right)^2 \int_0^R \left[\left(r^{-\frac{N_g}{2}} g(r) - \frac{2}{N_g-2} r^{1-\frac{N_g}{2}} g'(r)\right)^2 - \frac{v^2}{r^{N_g}}\right] r^{N_g-1}dr$$

$$= -\left(\frac{N_g-2}{2}\right)^2 \int_0^R \left(\frac{2}{N_g-2} r^{1-\frac{N_g}{2}} g'(r)\right)\left(2r^{-\frac{N_g}{2}} g(r) - \frac{2}{N_g-2} r^{1-\frac{N_g}{2}} g'\right) r^{N_g-1}dr$$

$$= -\left(\frac{N_g-2}{2}\right)^2 \int_0^R \frac{4}{N_g-2} r^{1-N_g} g'(r) g(r) r^{N_g-1}dr + \int_0^R r^{2-N_g}\left[g'(r)\right]^2 r^{N_g-1}dr$$

$$= -\left(\frac{N_g-2}{2}\right) \int_0^R \left(g^2(r)\right)' dr + \int_0^R \left[g'(r)\right]^2 rdr$$

We have

$$\int_0^R \left[g'(r)\right]^2 rdr \ge -\int_0^R \left[g(r)\right]^2 m(r)dr$$

where $m(r) = \frac{f(r) + rf''(r)}{f(r)}$. We obtain

$$\int_0^R \left[g'(r)\right]^2 rdr \ge -\int_0^R g_{n,i}^2(r) r^{N_g-2} m(r)dr$$

$$= -\int_0^R g_{n,i}^2(r) r^{N_g-2} \left(\frac{f'(r) + rf''(r)}{f(r)}\right)dr$$

$$= -\int_0^R g_{n,i}^2(r) \left(\frac{f'(r) + rf''(r)}{rf(r)}\right) r^{N_g-1}dr$$

So

$$\int_0^R \left(\left|\nabla_k f_{n,i}\right|^2 - \left(\frac{N_g-2}{2}\right)^2 \frac{f_{n,i}^2}{r^2}\right) r^{N_g-1}dr \ge -\int_0^R f_{n,i}^2(r) \left(\frac{f'(r) + rf''(r)}{rf(r)}\right) r^{N_g-1}dr$$

Finally

$$
\begin{aligned}
&\int_{\Omega}\left[\left|\nabla_k f\right|^2 - \Lambda\frac{f^2}{|x|^2}\right]d\mu_k(x)\\
&\geq \left|S^{N-1}\right|\sum_{n=0}^{\infty}\sum_{i=1}^{d(n)}\int_0^R\left(\left|\nabla_k f_{n,i}\right|^2-\left(\frac{N_g-2}{2}\right)^2\frac{f_{n,i}^2}{|x|^2}\right)r^{N_g-1}dr\\
&\geq \left|S^{N-1}\right|\sum_{n=0}^{\infty}\sum_{i=1}^{d(n)}\int_0^R f_{n,i}^2(r)\left(-\frac{f'(r)+rf''(r)}{rf(r)}\right)r^{N_g-1}dr\\
&= \int_{\Omega}P(|x|)f^2(x)d\mu_k(x)
\end{aligned}
$$

where

$$
P(|x|) = -\frac{f'(|x|)+rf''(|x|)}{|x|\,f(|x|)}.
$$

3. Conclusion

Main result. *Let* $N_g > 2$ *If* K *is a normalized HIP on* $(0,R)$, *we have*

$$
\int_{\Omega}\left|\nabla_k f\right|^2 d\mu_k(x) - \left(\frac{N_g-2}{2}\right)^2\int_{\Omega}\frac{f^2}{|x|^2}d\mu_k(x) \geq \int_{\Omega}P(|x|)f^2(x)d\mu_k(x).
$$

for any $u \in C_c^{\infty}(\Omega)$ *and* $N_g = N + 2g$.

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BẤT ĐẲNG THỨC HARDY CHỨA TOÁN TỬ DUNKL LIÊN KẾT VỚI DẠNG HI-POTENTIAL

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TÓM TẮT

Nghiên cứu thiết lập bất đẳng thức dạng Hardy liên quan đến các toán tử Dunkl và độ đo Dunkl liên kết với một HI-Potential. Bằng cách sử dụng khai triển h-harmonic của hàm $f \in L^2(\mu_k)$ thành $f(x) = f(rx) = \sum_{n=0}^{\infty}\sum_{i=1}^{d(n)} f_{n,i}(r)Y_i^n(x)$ trong đó Y_i^n là hàm riêng của toán tử $\Delta_{k,0}$ với giá trị riêng λ_n tương ứng. Kết hợp các phép biến đổi tích phân, các công thức tọa độ cầu, công thức tách biến, kết quả trong bài báo được thể hiện ở định lí 1. Kết quả này là sự tổng quát hóa kết quả đã được thiết lập bởi Ghoussoub và Moradifam (2013) về HIP. Kết quả này của chúng tôi đã tổng quát hóa bất đẳng thức dạng Hardy trong lí thuyết về toán tử Dunkl.

Từ khóa: hằng số tốt nhất; bất đẳng thức Hardy; HI-Potential