



Research Article

ON THE MORAVA K -THEORY OF THE REAL GRASSMANNIAN $Gr_2(\mathbb{R}^m)$

Phan Hoang Chon

Saigon University, Vietnam

Corresponding author: Phan Hoang Chon – Email: phchon@sgu.edu.vn

Received: October 3, 2025; Revised: November 3, 2025; Accepted: December 1, 2025

ABSTRACT

The main purpose of this paper is to determine an explicit $K(n)^*$ -basis for the n th Morava K -theory of the real Grassmannian $Gr_2(\mathbb{R}^m)$ for integers $n \geq 0$ and $m \geq 2$. Using Lenart's combinatorial formula, we compute the action of the primitive Milnor operation Q_n on Schur functions to determine the Q_n -homology of the mod 2 cohomology of $Gr_2(\mathbb{R}^m)$, thereby determining the $E_{2^{n+1}}$ -term of the Atiyah-Hirzebruch spectral sequence. We show that the size of the $E_{2^{n+1}}$ -term equals the lower bound for the size of $K(n)^*(Gr_2(\mathbb{R}^m))$ defined by Kuhn and Lloyd (2024); therefore, the spectral sequence collapses at the $E_{2^{n+1}}$ -term. This result provides foundational data for further investigations of Morava K -theory of real Grassmannians.

Keywords: Atiyah-Hirzebruch spectral sequence; Morava K -theory; Real Grassmannian

2020 Mathematics Subject Classification: Primary 55P47, 55Q45; Secondary 55S10, 55T15.

1. Introduction

Given a finite complex X , let $K(n)^*(X)$ denote the 2-local n th Morava K -theory of X . This is a graded vector space over the graded field $K(n)_* = \mathbb{F}_2[v_n^{\pm 1}]$ with $|v_n| = 2(2^n - 1)$. For $0 \leq d \leq m$, let $Gr_d(\mathbb{R}^m)$ be the real Grassmannian, namely the manifold of d -planes of $\mathbb{R}^m$. The purpose of this paper is to determine an explicit $K(n)^*$ -basis of $K(n)^*(Gr_2(\mathbb{R}^m))$ using the Atiyah-Hirzebruch spectral sequence (AHSS).

For a finite complex X , the AHSS that computes $K(n)^*(X)$ is given by

$$E_2^{s,t}(X) = H^s(X, K(n)^t) \Rightarrow K(n)^{s+t}(X),$$

where the differential $d_r : E_r^{s,t}(X) \rightarrow E_r^{s+r, t-r+1}(X)$. This is a spectral sequence of graded $K(n)^* = \mathbb{F}_2[v_n^{\pm 1}]$ -algebras, where v_n has cohomological degree $2 - 2^{n+1}$. The sparseness of

Cite this article as: Phan, H. C. (2026). On the morava K -theory of the real grassmannian $Gr_2(\mathbb{R}^m)$. *Ho Chi Minh City University of Education Journal of Science*, 23(8), 1797-1809. [https://doi.org/10.54607/hcmue.js.23.8.5287\(2026\)](https://doi.org/10.54607/hcmue.js.23.8.5287(2026))

the rows shows that the differential d_r vanishes unless $r = s(2^{n+1} - 2) + 1$ for some s . Yagita (1980) showed that the first nontrivial differential in the spectral sequence is $d_{2^{n+1}-1}$; moreover, for $x \in E_2^{*,0}(X) \cong H^*(X)$,

$$d_{2^{n+1}-1}(x) = Q_n(x) \otimes v_n.$$

Here, Q_n is the primitive Milnor operation, defined by

$$Q_0 = Sq^1, \quad Q_n = Q_{n-1}Sq^{2^n} + Sq^{2^n}Q_{n-1}.$$

Therefore, determining the $E_{2^{n+1}}$ -term of the AHSS is equivalent to determining the Q_n -homology of $H^*(X)$. Set

$$k_n(X) = \dim_{K(n)_*} K(n)^*(X),$$

the rank of $K(n)^*(X)$ over the graded field $K(n)_*$, and

$$k_{Q_n}(X) = \dim_{\mathbb{F}_2} H_*(H^*(X); Q_n),$$

the $\mathbb{F}_2$ -dimension of the Margolis homology of H^*X with respect to the Milnor primitive operation Q_n of the mod 2 Steenrod algebra A :

$$H^*(X; Q_n) := \text{Ker}[Q_n : H^*X \rightarrow H^*X] / \text{Im}[Q_n : H^*X \rightarrow H^*X].$$

Here and below, H^*X denotes the mod 2 cohomology of X .

Using chromatic fixed-point theory, Kuhn and Lloyd (2024) proved the following result.

Theorem 1.1. (Kuhn and Lloyd (2024)). (1) If $m \leq 2^{n+1}$, then $k_n(Gr_d(\mathbb{R}^m)) = \binom{m}{d}$.

(2) If $m = 2^{n+1} - \epsilon + 2\ell$, $\ell \geq 1$, $\epsilon = 0, 1$, then

$$k_{Q_n}(Gr_d(\mathbb{R}^m)) \geq k_n(Gr_d(\mathbb{R}^m)) \geq \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{2^{n+1} - \epsilon}{d - 2i} \binom{\ell}{i}.$$

They also conjectured that the inequalities in Theorem 1.1 are always equalities:

Conjecture 1.2. (Kuhn and Lloyd (2024)). If $m = 2^{n+1} - \epsilon + 2\ell$, $\ell \geq 1$, $\epsilon = 0, 1$, then

$$k_{Q_n}(Gr_d(\mathbb{R}^m)) = k_n(Gr_d(\mathbb{R}^m)) = \sum_{i=0}^{\lfloor d/2 \rfloor} \binom{2^{n+1} - \epsilon}{d - 2i} \binom{\ell}{i}.$$

It is straightforward to verify that Conjecture 1.2 holds for $d = 1$. Kuhn and Lloyd (2024) proved that Conjecture 1.2 holds for $d = 2$; however, they did not identify an explicit basis for $K(n)^*(Gr_2(\mathbb{R}^m))$.

In Phan and Nguyen (2026), we verified that Conjecture 1.2 holds for $d = 3$. To treat the case $d > 3$ within our framework, we require an explicit $K(n)^*$ -basis of $K(n)^*(Gr_2(\mathbb{R}^m))$ as well as an $\mathbb{F}_2$ -basis of Q_n -homology of $H^*(Gr_2(\mathbb{R}^m))$. This need motivates the present work. Although the size of $K(n)^*(Gr_2(\mathbb{R}^m))$ is known (see Kuhn & Lloyd, 2024), the

problem of determining a $K(n)^*$ -basis of $K(n)^*(Gr_2(\mathbb{R}^m))$ remains nontrivial. To solve this problem, we compute the kernel of the connecting homomorphism δ in the long exact sequence (2.4) (see below) and determine an $\mathbb{F}_2$ -basis of $H^*(Gr_2(\mathbb{R}^m); \mathbb{Q}_n)$ when m is odd.

The paper is organized into three sections. Section 2 reviews the cohomology of real Grassmannians, Lenart's formula, and a short exact sequence needed for the proofs. Section 3 presents the main results.

2. Preliminaries

To fix notation, we first recall the description of the mod 2 cohomology of the real Grassmannian as a quotient of the ring of symmetric polynomials.

Let $Gr_d(\mathbb{R}^{d+k})$ be the real Grassmann manifold of d -planes in $\mathbb{R}^{d+k}$. The natural filtration

$$Gr_d(\mathbb{R}^d) \subset Gr_d(\mathbb{R}^{d+1}) \subset \cdots \subset Gr_d(\mathbb{R}^{d+k}) \subset \cdots$$

defines the infinite real Grassmann manifold as a colimit $Gr_d(\mathbb{R}^\infty) := \text{colim}_k Gr_d(\mathbb{R}^{d+k})$.

Let γ_1 be the canonical line bundle over the infinite real projective space $\mathbb{R}P^\infty$, and let $\gamma_1^{\times d}$ be the d -plane bundle over $(\mathbb{R}P^\infty)^d$. The bundle $\gamma_1^{\times d}$ is classified by a map $g : (\mathbb{R}P^\infty)^d \rightarrow Gr_d(\mathbb{R}^\infty)$. This map induces a monomorphism

$$g^* : H^*Gr_d(\mathbb{R}^\infty) \rightarrow H^*(\mathbb{R}P^\infty)^d$$

whose image is the subring of symmetric polynomials in $H^*(\mathbb{R}P^\infty)^d \cong \mathbb{F}_2[x_1, \dots, x_d]$. Moreover, g^* sends the i -th Stiefel-Whitney class $w_i = w_i(\gamma_d)$ of the canonical d -plane bundle over $Gr_d(\mathbb{R}^\infty)$ to the i -th elementary symmetric polynomial e_i in $x_1, \dots, x_d$. The mod 2 cohomology of $Gr_d(\mathbb{R}^\infty)$ is therefore the polynomial algebra described by Milnor and Stasheff (1974):

$$H^*Gr_d(\mathbb{R}^\infty) \cong \mathbb{F}_2[w_1, \dots, w_d].$$

The dual classes $\bar{w}_1, \bar{w}_2, \dots$ are defined by the equation

$$(1 + w_1 + \cdots + w_d)(1 + \bar{w}_1 + \bar{w}_2 + \cdots) = 1, \quad (2.1)$$

and this allows one to write each $\bar{w}_k$ as a polynomial in $w_1, \dots, w_d$. The class $\bar{w}_k$ can be interpreted as $w_k(\gamma_d^\perp)$ where $\gamma_d^\perp$ is the k -dimensional bundle complementary to γ_d . For dimensional reasons, in $H^*Gr_d(\mathbb{R}^{d+k})$, $\bar{w}_j = 0$ for all $j > k$. Moreover, the inclusion $Gr_d(\mathbb{R}^{d+k}) \rightarrow Gr_d(\mathbb{R}^\infty)$ induces a ring epimorphism

$$H^*Gr_d(\mathbb{R}^\infty) \twoheadrightarrow H^*Gr_d(\mathbb{R}^{d+k})$$

whose kernel is given by the ideal $J(d, k) = (\bar{w}_j \mid j > k)$.

We recall the cell structure of $Gr_d(\mathbb{R}^{d+k})$ as described in Milnor and Stasheff (1974) (see also Hiller (1982)). A Schubert symbol $\lambda = (\lambda_1, \dots, \lambda_d)$ of $Gr_d(\mathbb{R}^{d+k})$ is a sequence of integers

$$k \geq \lambda_1 \geq \dots \geq \lambda_d \geq 0.$$

The weight of λ is denoted by $|\lambda|$ and is defined by

$$|\lambda| = \lambda_1 + \dots + \lambda_d.$$

Thus, a Schubert symbol λ is a partition contained in a $d \times k$ grid when depicted as a Young diagram.

Each Schubert symbol λ determines a Schubert cell Ω_λ of dimension $|\lambda|$ in $Gr_d(\mathbb{R}^{d+k})$, defined by

$$\Omega_\lambda = \{V \in Gr_d(\mathbb{R}^{d+k}) : \dim(V \cap \mathbb{R}^{i+\lambda_{d+1-i}}) = i, 1 \leq i \leq d+k\}.$$

The duals of the Schubert cells form a $\mathbb{Z}_2$ -basis of $H^*Gr_d(\mathbb{R}^{d+k})$. Moreover, the dual of Ω_λ , denoted by $s_\lambda \in H^{|\lambda|}Gr_d(\mathbb{R}^{d+k})$, corresponds to the Schur symmetric function associated with the partition λ (see Macdonald (1995)). The class w_i in the Borel presentation corresponds to the elementary symmetric function $s_{(1^i)} = e_i$ and the dual class $\bar{w}_j$ to $s_{(j)} = h_j$, the j -th complete symmetric polynomial. The ring structure of $H^*Gr_d(\mathbb{R}^{d+k})$ in this basis is described by the Littlewood-Richardson rule from the theory of symmetric functions.

Recall that the Milnor operation Q_n in the Steenrod algebra A is recursively defined by

$$Q_0 = Sq^1; \quad Q_{n+1} = Q_n Sq^{2^n} + Sq^{2^n} Q_n, \quad n \geq 0.$$

To state Lenart's (1998) formula for the action of Q_n on Schur symmetric functions, we need some combinatorial definitions.

Given a Young diagram λ contained in another Young diagram μ , one can form the complement of λ in μ , denoted μ/λ and called a skew diagram. The content of a box b of a Young diagram μ in row i and column j is defined by $c(b) = j - i$. For a box b in the skew diagram μ/λ , its content is the content of b embedded in μ .

A skew shape is connected if any two boxes in the diagram can be joined by a sequence of edge-adjacent boxes. A skew shape that does not contain a 2×2 -block of boxes is called a broken border strip. A connected broken border strip is called a border strip. If λ is a broken border strip, $cc(\lambda)$ denotes the number of connected components of λ . A sharp corner of a broken border strip is a box with no northern, western, or northwestern neighbor. A dull corner of a broken strip is a box with both northern and western neighbors but no northwestern neighbor. Let $C(\mu/\lambda)$ denote the set of sharp and dull corners of μ/λ .

We are now ready to state the formula in Lenart (1998, Proposition 5.1) for the action of the Milnor operation on a Schubert basis class:

Proposition 2.1. *We have*

$$Q_n(s_\lambda) = \sum_{\mu \supset \lambda : |\mu| - |\lambda| = 2^{n+1} - 1; cc(\mu/\lambda) \leq 2} d_{\lambda\mu} s_\mu, \quad (2.2)$$

where μ/λ must be a broken border strip and

$$d_{\lambda\mu} = \begin{cases} \sum_{b \in C(\mu/\lambda)} c(b), & \text{if } \mu/\lambda \text{ is connected,} \\ 1, & \text{if } \mu/\lambda \text{ is disconnected.} \end{cases} \quad (2.3)$$

Recall that the Schubert basis of $H^*Gr_d(\mathbb{R}^m)$ consists of the classes s_λ satisfying $m-d \geq \lambda_1 \geq \dots \geq \lambda_d \geq 0$. Let $i: Gr_d(\mathbb{R}^{m-1}) \rightarrow Gr_d(\mathbb{R}^m)$ denote the map induced by the natural inclusion $\mathbb{R}^{m-1} \rightarrow \mathbb{R}^m$. In cohomology, this map induces a surjection

$$i^*: H^*Gr_d(\mathbb{R}^m) \twoheadrightarrow H^*Gr_d(\mathbb{R}^{m-1})$$

which, in the Schubert basis, is given by

$$i^*(s_{(\lambda_1, \dots, \lambda_d)}) = \begin{cases} 0, & \text{if } \lambda_1 = m-d, \\ s_{(\lambda_1, \dots, \lambda_d)}, & \text{if } \lambda_1 \leq m-d-1. \end{cases}$$

Proposition 2.2. *There is a short exact sequence of A -modules*

$$0 \rightarrow C_{d,m} \rightarrow H^*Gr_d(\mathbb{R}^m) \xrightarrow{i^*} H^*Gr_d(\mathbb{R}^{m-1}) \rightarrow 0$$

where the kernel $C_{d,m}$ of i^* has a linear basis consisting of the classes $s_{(\lambda_1, \dots, \lambda_d)}$ with $m-d = \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_d \geq 0$.

Remark 2.3. Let $C_d(\mathbb{R}^m)$ denote the cofibre of the inclusion $i: Gr_d(\mathbb{R}^{m-1}) \rightarrow Gr_d(\mathbb{R}^m)$. Kuhn and Lloyd (2024) showed that this cofibre is homeomorphic to the Thom space $Gr_{d-1}(\mathbb{R}^{m-1})^{\gamma_{d-1}^\perp}$, where $\gamma_{d-1}^\perp$ is the bundle complementary to the canonical bundle γ_{d-1} over $Gr_{d-1}(\mathbb{R}^{m-1})$. The cofibre sequence $Gr_d(\mathbb{R}^{m-1}) \xrightarrow{i} Gr_d(\mathbb{R}^m) \xrightarrow{p} C_d(\mathbb{R}^m)$ thus induces the short exact sequence in Proposition 2.2. In other words, $C_{d,m} \cong \tilde{H}^*C_d(\mathbb{R}^m)$.

The short exact sequence in Proposition 2.2 induces a long exact sequence in Q_n -homology

$$\begin{aligned} \dots \rightarrow H^*(C_{d,m}; Q_n) &\xrightarrow{p^*} H^*(Gr_d(\mathbb{R}^m); Q_n) \xrightarrow{i^*} H^*(Gr_d(\mathbb{R}^{m-1}); Q_n) \\ &\xrightarrow{\delta} H^{*+2^{n+1}-1}(C_d(\mathbb{R}^m); Q_n) \rightarrow \dots \end{aligned} \quad (2.4)$$

3. Main results

In this section, we use the long exact sequence (2.4) to describe an $\mathbb{F}_2$ -basis of Q_n -homology of $H^*Gr_2(\mathbb{R}^m)$, and thereby obtain a $K^*(n)$ -basis of $K^*(n)(Gr_2(\mathbb{R}^m))$

To determine an $\mathbb{F}_2$ -basis of $H^*(Gr_2(\mathbb{R}^{2+k}); Q_n)$, we first need an $\mathbb{F}_2$ -basis of $\tilde{H}^*(C_2(\mathbb{R}^{2+k}); Q_n)$. Kuhn and Lloyd's result gives the following description of $\tilde{H}^*(C_2(\mathbb{R}^{2+k}); Q_n)$.

Proposition 3.1. (1) In $\tilde{H}^*(C_2(\mathbb{R}^{2+k}); \mathbb{F}_2)$,

$$Q_n(s_{(k,i)}) = \begin{cases} s_{(k, 2^{n+1}-1+i)}, & \text{if } i \text{ is even and } k-i \leq 2^{n+1}-1, \\ 0, & \text{otherwise.} \end{cases}$$

(2) If $2+k \leq 2^{n+1}$, then Q_n acts as zero on $H^*(C_2(\mathbb{R}^{2+k}); \mathbb{F}_2)$. Thus $\tilde{k}_{Q_n}(C_2(\mathbb{R}^{2+k})) = k+1$.

(3) If $2+k > 2^{n+1}$ and k is even, then the classes $\{s_{(k, 2^{j-1})} : 1 \leq j \leq 2^n - 1\}$ and $\{s_{(k, k-2j)} : 0 \leq j \leq 2^n - 1\}$ represent the Q_n -homology classes. Thus, $\tilde{k}_{Q_n}(C_2(\mathbb{R}^{2+k})) = 2^{n+1} - 1$.

(4) If $2+k > 2^{n+1}$ and k is odd, then the classes $\{s_{(k, 2^{j-1})} : 1 \leq j \leq 2^n - 1\}$ and $\{s_{(k, k-1-2j)} : 0 \leq j \leq 2^n - 2\}$ represent the Q_n -homology classes. Thus, $\tilde{k}_{Q_n}(C_2(\mathbb{R}^{2+k})) = 2^{n+1} - 2$.

We now compute a basis of $H^*(Gr_2(\mathbb{R}^{2+k}); Q_n)$.

Theorem 3.2. (1) If $2+k \leq 2^{n+1}$, $H^*(Gr_2(\mathbb{R}^{2+k}); Q_n)$ is isomorphic to $H^*(Gr_2(\mathbb{R}^{2+k}); \mathbb{F}_2)$.

(2) If $2+k = 2^{n+1} - 1 + 2\ell$, then $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2\ell}); Q_n)$ has an $\mathbb{F}_2$ -basis represented by

$$\mathcal{B}_{2^{n+1}-1+2\ell};$$

(3) If $2+k = 2^{n+1} + 2\ell$, then $H^*(Gr_2(\mathbb{R}^{2^{n+1}+2\ell}); Q_n)$ has an $\mathbb{F}_2$ -basis represented by

$$\mathcal{B}_{2^{n+1}+2\ell} = \mathcal{B}_{2^{n+1}-1+2\ell} \cup \mathcal{C}_{2^{n+1}+2\ell},$$

where

$$\mathcal{C}_{2^{n+1}+2\ell} = \{s_{(2^{n+1}-2+2\ell, 2j-1)} \mid 1 \leq j \leq 2^n - 1\} \cup \{s_{(2^{n+1}-2+2\ell, 2^{n+1}-2+2\ell-2j)} \mid 0 \leq j \leq 2^n - 1\},$$

and, for $1 \leq \ell \leq 2^n - 1$,

$$\mathcal{B}_{2^{n+1}-1+2\ell} = \{s_{(i,j)} \mid 2\ell \leq j \leq i \leq 2(2^n - 1)\} \cup \{s_{(2j, 2j)} \mid 0 \leq j \leq \ell - 1\}$$

$$\cup \{s_{(2^{n+1}-2+2(i+j), 2i+1)} + s_{(2^{n+1}-1+2(i+j), 2i)} \mid 0 \leq i+j \leq \ell - 1\}$$

$$\cup \{s_{(2^{n+1}-2+2i, 2(i+j)+1)} + s_{(2^{n+1}-1+2(i+j), 2i)} \mid 1 \leq i+j \leq \ell - 1; j \geq 1\}$$

$$\cup \{s_{(2j-2, 2i+1)} + s_{(2j-1, 2i)} \mid 0 \leq i \leq \ell - 1; \ell + 1 \leq j \leq 2^n - 1\}$$

$$\cup \{s_{(2^{n+1}-3+2(i-j), 2i+1)} + s_{(2^{n+1}-2+2(i-j), 2i)} \mid 0 \leq i \leq \ell - 1; i \leq j \leq 2^n - 2\}$$

$$\cup \{s_{(2^{n+1}+2i, 2j-1)} \mid 0 \leq i \leq \ell - 2; \ell + 1 \leq j \leq 2^n - 1\}$$

$$\cup \{s_{(2^{n+1}+2i, 2^{n+1}+2i-2j)} \mid 0 \leq i \leq \ell - 2; 0 \leq j \leq 2^n - \ell + i\},$$

for $\ell = 2^n + s$ with $0 \leq s \leq 2^n - 2$,

$$\mathcal{B}_{2^{n+1}-1+2(2^n+s)} = \{s_{(2j, 2j)} \mid 0 \leq j \leq 2^n + s - 1\}$$

$$\cup \{s_{(2^{n+1}-2+2(i+j), 2i+1)} + s_{(2^{n+1}-1+2(i+j), 2i)} \mid s+1 \leq i+j \leq 2^n - 2\}$$

$$\cup \{s_{(2^{n+1}-2+2i, 2j-1)} + s_{(2^{n+1}-1+2i, 2j-2)} \mid 2^n - 1 \leq i \leq 2^n - 1 + s; 1 \leq j \leq 2^n - 1\}$$

$$\cup \{s_{(2^{n+1}-2+2i, 2(i+j)+1)} + s_{(2^{n+1}-1+2(i+j), 2i)} \mid i + j \leq 2^n - 2; j \geq 1; i \geq s + 1\}$$

$$\cup \{s_{(2^{n+1}-3+2(i-j), 2i+1)} + s_{(2^{n+1}-2+2(i-j), 2i)} \mid 0 \leq i \leq 2^n - 2; i \leq j \leq 2^n - 2\}$$

$$\cup \{s_{(2^{n+1}-2+2i, 2^{n+1}-2+2i-2j)} \mid s + 1 \leq i \leq 2^n - 1 + s; 0 \leq j \leq i - (s + 1)\},$$

for $\ell = 2^n + s$ with $s \geq 2^n - 1$

$$\mathcal{B}_{2^{n+1}-1+2(2^n+s)} = \{s_{(2j, 2j)} \mid 0 \leq j \leq 2^n + s - 1\}$$

$$\cup \{s_{(2^{n+1}-2+2i, 2j-1)} + s_{(2^{n+1}-1+2i, 2j-2)} \mid s + 1 \leq i \leq 2^n - 1 + s; 1 \leq j \leq 2^n - 1\}$$

$$\cup \{s_{(2^{n+1}-3+2(i-j), 2i+1)} + s_{(2^{n+1}-2+2(i-j), 2i)} \mid 0 \leq i \leq 2^n - 2; i \leq j \leq 2^n - 2\}$$

$$\cup \{s_{(2^{n+1}-2+2i, 2^{n+1}-2+2i-2j)} \mid s + 1 \leq i \leq 2^n - 1 + s; 0 \leq j \leq i - (s + 1)\}.$$

Part (1) of the theorem follows from the fact that, for $d + k \leq 2^{n+1}$, Q_n acts as zero on $H^*(Gr_d(\mathbb{R}^{d+k}); \mathbb{F}_2)$ (see Kuhn & Lloyd, 2024).

For $d = 2$ with $2 + k$ even, the connecting homomorphism δ of (2.4) is zero (Kuhn & Lloyd, 2024). Therefore, Proposition 3.1 implies the final claim.

We prove part (2) by induction on ℓ . For the inductive step at $\ell + 1$, we use the hypothesis in part (3) for ℓ together with Proposition 3.1. We then compute an $\mathbb{F}_2$ -basis of the kernel of the connecting homomorphism δ of (2.4), thereby obtaining the required basis. To compute the kernel of δ , we use Lenart's formulas (2.2)-(2.3) to compute the action of Q_n on elements of $H^*(Gr_d(\mathbb{R}^m))$. In general, this computation is complicated because condition (2.3) is difficult to verify. In our case, however, the following observation simplifies the computation.

Let $x = \sum_{\lambda} s_{\lambda} \in H^*(Gr_d(\mathbb{R}^m))$ be a cycle representing a nontrivial homology class in $H^*(Gr_d(\mathbb{R}^m); Q_n)$. Then x can be regarded as its own preimage in $H^*(Gr_d(\mathbb{R}^{m+1}))$ under i^* . In $H^*(Gr_d(\mathbb{R}^{m+1}))$,

$$Q_n(x) = \sum_{m-d \geq \mu_1 \geq \dots \geq \mu_d} d_{\mu} s_{\mu} + \sum_{m+1-d = \nu_1 \geq \dots \geq \nu_d} d_{\nu} s_{\nu},$$

where d_{μ} and d_{ν} belong to $\mathbb{F}_2$. Because $Q_n(x) = 0$ in $H^*(Gr_d(\mathbb{R}^m))$, the first sum in the formula above vanishes; it therefore remains to compute d_{ν} in the second sum. This makes the computation in this case much easier than in the general case.

For example, let $x = s_{(2^{n+1}-2+2\ell, 2j-1)} \in H^*(Gr_2(\mathbb{R}^{2^{n+1}+2\ell}))$ be a cycle representing a nontrivial homology class in $H^*(Gr_2(\mathbb{R}^{2^{n+1}+2\ell}); Q_n)$ for $\ell \geq 1$ and $1 \leq j \leq 2^n - 1$. Then x can

be regarded as its own preimage in $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2(\ell+1)}))$ under the map i . Because $Q_n(x) = 0$ in $H^*(Gr_2(\mathbb{R}^{2^{n+1}+2\ell}))$, we have the following identity in $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2(\ell+1)}))$:

$$Q_n(x) = d_t s_{(2^{n+1}-3+2(\ell+1),t)},$$

where $d_t \in \mathbb{F}_2$ and $(2^{n+1}-3+2(\ell+1),t)/(2^{n+1}-2+2\ell,2j-1)$ must be a broken border strip containing exactly $2^{n+1}-1$ boxes. We can directly check that

- if $j \geq \ell + 2$, then $t > 2^{n+1}-3+2(\ell+1)$, which implies $Q_n(x) = 0$;
- if $j \leq \ell$, then $(2^{n+1}-3+2(\ell+1),t)/(2^{n+1}-2+2\ell,2j-1)$ is a disconnected broken border strip, which implies $d_t = 1$ (see condition (2.3));
- if $j = \ell + 1$, then $(2^{n+1}-3+2(\ell+1),t)/(2^{n+1}-2+2\ell,2j-1)$ is a connected broken border strip with two sharp corners of even content and a dull corner of odd content; therefore, $d_t = 1$.

Hence,

$$Q_n(x) = 0, \ell + 2 \leq j \leq 2^n - 1;$$

$$Q_n(x) = s_{(2^{n+1}-3+2(\ell+1),2^{n+1}-3+2j)}, 1 \leq j \leq \ell + 1.$$

We use this strategy throughout to compute the image of Q_n in the proof of (2) and therefore omit the repeated details.

Proof of (2). The proof is divided into three parts corresponding to the three cases for ℓ .

Proof of the case $1 \leq \ell \leq 2^n - 1$. From (1), $k_{Q_n}(Gr_2(\mathbb{R}^{2^{n+1}})) = \binom{2^{n+1}}{2}$ and the set

$$\mathcal{B}_{2^{n+1}} = \{s_{(i,j)} \mid 0 \leq j \leq i \leq 2^{n+1} - 2\}$$

represents an $\mathbb{F}_2$ -basis of $H^*(Gr_2(\mathbb{R}^{2^{n+1}}); Q_n)$. Since $\mathcal{B}_{2^{n+1}}$ can be regarded as its own preimage in $H^*(Gr_2(\mathbb{R}^{2^{n+1}+1}))$ under i , in $H^*(Gr_2(\mathbb{R}^{2^{n+1}+1}))$, we have

$$Q_n(s_{(i,1)}) = s_{(2^{n+1}-1,i+1)}, \text{ for } 1 \leq i \leq 2^{n+1} - 2;$$

$$Q_n(s_{(i)}) = s_{(2^{n+1}-1,i)}, \text{ for } 1 \leq i \leq 2^{n+1} - 2;$$

and Q_n sends all other elements of $\mathcal{B}_{2^{n+1}}$ to zero. Therefore, since $\delta([x]) = [Q_n(i^{-1}(x))]$,

$$\delta([s_{(2j-2,1)}]) = s_{(2^{n+1}-1,2j-1)}, \text{ for } 2 \leq j \leq 2^n - 1;$$

$$\delta([s_{(2^{n+1}-2,1)}]) = [Q_n(s_{(2^{n+1}-1)})];$$

$$\delta([s_{(2^{n+1}-3-2j,1)}]) = [s_{(2^{n+1}-1,2^{n+1}-2-2j)}], \text{ for } 0 \leq j \leq 2^n - 2;$$

$$\delta([s_{(2j-1)}]) = [s_{(2^{n+1}-1,2j-1)}], \text{ for } 1 \leq j \leq 2^n - 1;$$

$$\delta([s_{(2^{n+1}-2-2j)}]) = [s_{(2^{n+1}-1,2^{n+1}-2-2j)}], \text{ for } 0 \leq j \leq 2^n - 2;$$

and δ sends all other elements to zero. Thus, δ is an epimorphism, and the set

$$\begin{aligned}\mathcal{B}_{2^{n+1}+1} &= \{s_{(i,j)} : 2 \leq j \leq i \leq 2^{n+1}-2\} \cup \{s_{(2j,2j)} \mid 0 \leq j \leq 0\} \\ &\cup \{s_{(2^{n+1}-2+2j,1)} + s_{(2^{n+1}-1+2j)} \mid 0 \leq j \leq 0\} \\ &\cup \{s_{(2j-2,1)} + s_{(2j-1)} \mid 2 \leq j \leq 2^n-1\} \\ &\cup \{s_{(2^{n+1}-3-2j,1)} + s_{(2^{n+1}-2-2j)} \mid 0 \leq j \leq 2^n-2\}\end{aligned}$$

represents an $\mathbb{F}_2$ -basis of the kernel of δ , and hence of $H^*(Gr_2(\mathbb{R}^{2^{n+1}+1}); Q_n)$. Hence, part (2) holds for $\ell = 1$.

Assume that for $1 \leq \ell \leq 2^n - 2$, $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2\ell}); Q_n)$ has an $\mathbb{F}_2$ -basis represented by

$$\begin{aligned}\mathcal{B}_{2^{n+1}-1+2\ell} &= \{s_{(i,j)} \mid 2\ell \leq j \leq i \leq 2(2^n-1)\} \cup \{s_{(2j,2j)} \mid 0 \leq j \leq \ell-1\} \\ &\cup \{s_{(2^{n+1}-2+2(i+j),2i+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid 0 \leq i+j \leq \ell-1\} \\ &\cup \{s_{(2^{n+1}-2+2i,2(i+j)+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid 1 \leq i+j \leq \ell-1; j \geq 1\} \\ &\cup \{s_{(2j-2,2i+1)} + s_{(2j-1,2i)} \mid 0 \leq i \leq \ell-1; \ell+1 \leq j \leq 2^n-1\} \\ &\cup \{s_{(2^{n+1}-3+2(i-j),2i+1)} + s_{(2^{n+1}-2+2(i-j),2i)} \mid 0 \leq i \leq \ell-1; i \leq j \leq 2^n-2\} \\ &\cup \{s_{(2^{n+1}+2i,2j-1)} \mid 0 \leq i \leq \ell-2; \ell+1 \leq j \leq 2^n-1\} \\ &\cup \{s_{(2^{n+1}+2i,2^{n+1}+2i-2j)} \mid 0 \leq i \leq \ell-2; 0 \leq j \leq 2^n-\ell+i\},\end{aligned}$$

By part (3), $H^*(Gr_2(\mathbb{R}^{2^{n+1}+2\ell}); Q_n)$ has an $\mathbb{F}_2$ -basis represented by

$$\mathcal{B}_{2^{n+1}+2\ell} = \mathcal{B}_{2^{n+1}-1+2\ell} \cup \mathcal{C}_{2^{n+1}+2\ell},$$

where

$$\mathcal{C}_{2^{n+1}+2\ell} = \{s_{(2^{n+1}+2(\ell-1),2j-1)} \mid 1 \leq j \leq 2^n-1\} \cup \{s_{(2^{n+1}+2(\ell-1),2^{n+1}+2(\ell-1)-2j)} \mid 0 \leq j \leq 2^n-1\}.$$

Applying the preceding observation, we obtain, in $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2(\ell+1)}))$,

$$\begin{aligned}Q_n(s_{(i,2\ell+1)}) &= s_{(2^{n+1}-1+2\ell,i+1)}, \text{ for } 2\ell+1 \leq i \leq 2(2^n-1); \\ Q_n(s_{(i,2\ell)}) &= s_{(2^{n+1}-1+2\ell,i)}, \text{ for } 2\ell+1 \leq i \leq 2(2^n-1); \\ Q_n(s_{(2\ell,2i+1)} + s_{(2\ell+1,2i)}) &= s_{(2^{n+1}-1+2\ell,2i+1)}, \text{ for } 0 \leq i \leq \ell-1; \\ Q_n(s_{(2^{n+1}+2i,2\ell+1)}) &= s_{(2^{n+1}-1+2\ell,2^{n+1}+2i+1)}, \text{ for } 0 \leq i \leq \ell-2; \\ Q_n(s_{(2^{n+1}+2i,2\ell)}) &= s_{(2^{n+1}-1+2\ell,2^{n+1}+2i)}, \text{ for } 0 \leq i \leq \ell-2; \\ Q_n(s_{(2^{n+1}+2(\ell-1),2j-1)}) &= s_{(2^{n+1}-1+2\ell,2^{n+1}-3+2j)}, \text{ for } 1 \leq j \leq \ell+1; \\ Q_n(s_{(2^{n+1}+2(\ell-1),2\ell)}) &= s_{(2^{n+1}-1+2\ell,2^{n+1}-2+2\ell)};\end{aligned}$$

and Q_n sends all other elements of $\mathcal{B}_{2^{n+1}+2\ell}$ to zero. Therefore,

$$\delta([s_{(2^{n+1}+2(\ell-1),2j-1)}]) = [s_{(2^{n+1}-1+2\ell,2^{n+1}-3+2j)}] = [Q_n(s_{(2^{n+1}-1+2\ell,2(j-1))})], \text{ for } 1 \leq j \leq \ell+1;$$

$$\begin{aligned}
\delta([s_{(2^{n+1}-2, 2\ell+1)}]) &= [s_{(2^{n+1}-1+2\ell, 2^{n+1}-1)}] = [Q_n(s_{(2^{n+1}-1+2\ell)})]; \\
\delta([s_{(2^{n+1}+2i, 2\ell+1)}]) &= [s_{(2^{n+1}-1+2\ell, 2^{n+1}+2i+1)}] = [Q_n(s_{(2^{n+1}-1+2\ell, 2(i+1))})], \text{ for } 0 \leq i \leq \ell-2; \\
\delta([s_{(2j-2, 2\ell+1)}]) &= [s_{(2^{n+1}-1+2\ell, 2j-1)}], \text{ for } \ell+2 \leq j \leq 2^n-1; \\
\delta([s_{(2^{n+1}-3+2(\ell-j), 2\ell+1)}]) &= [s_{(2^{n+1}-1+2\ell, 2^{n+1}-2+2(\ell-j))}], \text{ for } \ell \leq j \leq 2^n-2; \\
\delta([s_{(2j-1, 2\ell)}]) &= [s_{(2^{n+1}-1+2\ell, 2j-1)}], \text{ for } \ell+1 \leq j \leq 2^n-1; \\
\delta([s_{(2^{n+1}-2+2(\ell-j), 2\ell)}]) &= [s_{(2^{n+1}-1+2\ell, 2^{n+1}-2+2(\ell-j))}], \text{ for } \ell \leq j \leq 2^n-2; \\
\delta([s_{(2\ell, 2i+1)} + s_{(2\ell+1, 2i)}]) &= [s_{(2^{n+1}-1+2\ell, 2i+1)}], \text{ for } 0 \leq i \leq \ell-1; \\
\delta([s_{(2^{n+1}+2i, 2\ell)}]) &= [s_{(2^{n+1}-1+2\ell, 2^{n+1}+2i)}], \text{ for } 0 \leq i \leq \ell-2; \\
\delta([s_{(2^{n+1}+2(\ell-1), 2\ell)}]) &= [s_{(2^{n+1}-1+2\ell, 2^{n+1}-2+2\ell)}];
\end{aligned}$$

and δ sends all other elements to zero. Thus, δ is an epimorphism and the set

$$\begin{aligned}
\mathcal{B}_{2^{n+1}-1+2(\ell+1)} &= \{s_{(i,j)} \mid 2(\ell+1) \leq j \leq i \leq 2(2^n-1)\} \cup \{s_{(2j, 2j)} \mid 0 \leq j \leq \ell\} \\
&\cup \{s_{(2^{n+1}-2+2(i+j), 2i+1)} + s_{(2^{n+1}-1+2(i+j), 2i)} \mid 0 \leq i+j \leq \ell\} \\
&\cup \{s_{(2^{n+1}-2+2i, 2(i+j)+1)} + s_{(2^{n+1}-1+2(i+j), 2i)} \mid 1 \leq i+j \leq \ell; j \geq 1\} \\
&\cup \{s_{(2j-2, 2i+1)} + s_{(2j-1, 2i)} \mid 0 \leq i \leq \ell; \ell+2 \leq j \leq 2^n-1\} \\
&\cup \{s_{(2^{n+1}-3+2(i-j), 2i+1)} + s_{(2^{n+1}-2+2(i-j), 2i)} \mid 0 \leq i \leq \ell; i \leq j \leq 2^n-2\} \\
&\cup \{s_{(2^{n+1}+2i, 2j-1)} \mid 0 \leq i \leq \ell-1; \ell+2 \leq j \leq 2^n-1\} \\
&\cup \{s_{(2^{n+1}+2i, 2^{n+1}+2i-2j)} \mid 0 \leq i \leq \ell-1; 0 \leq j \leq 2^n-(\ell+1)+i\}
\end{aligned}$$

represents an $\mathbb{F}_2$ -basis of the kernel of δ , and hence an $\mathbb{F}_2$ -basis of $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2(\ell+1)}); Q_n)$.

Thus, part (2) holds for $1 \leq \ell \leq 2^n-1$.

Proof of the case $\ell \geq 2^n$. In this case, set $\ell = 2^n + s$ for $s \geq 0$. By the preceding result for $\ell = 2^n-1$ and part (3), $H^*(Gr_2(\mathbb{R}^{2^{n+1}+2(2^n-1)}); Q_n)$ has an $\mathbb{F}_2$ -basis represented by

$$\begin{aligned}
\mathcal{B}_{2^{n+1}+2(2^n-1)} &= \{s_{(2j, 2j)} \mid 0 \leq j \leq 2^n-1\} \\
&\cup \{s_{(2^{n+1}-2+2(i+j), 2i+1)} + s_{(2^{n+1}-1+2(i+j), 2i)} \mid 0 \leq i+j \leq 2^n-2\} \\
&\cup \{s_{(2^{n+1}-2+2i, 2(i+j)+1)} + s_{(2^{n+1}-1+2(i+j), 2i)} \mid 1 \leq i+j \leq 2^n-2; j \geq 1\} \\
&\cup \{s_{(2^{n+1}-3+2(i-j), 2i+1)} + s_{(2^{n+1}-2+2(i-j), 2i)} \mid 0 \leq i \leq 2^n-2; i \leq j \leq 2^n-2\} \\
&\cup \{s_{(2^{n+1}+2i, 2^{n+1}+2i-2j)} \mid 0 \leq i \leq 2^n-3; 0 \leq j \leq i+1\} \\
&\cup \{s_{(2^{n+1}-2+2(2^n-1), 2j-1)} \mid 1 \leq j \leq 2^n-1\} \cup \{s_{(2^{n+1}-2+2(2^n-1), 2^{n+1}-2+2(2^n-1)-2j)} \mid 0 \leq j \leq 2^n-1\}.
\end{aligned}$$

For $\ell = 2^n$, in $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2 \cdot 2^n}); \mathbb{F}_2)$, we have

$$\begin{aligned} Q_n(s_{(2^{n+1}-2,1)} + s_{(2^{n+1}-1)}) &= s_{(2^{n+1}-1+2(2^n-1),1)}; \\ Q_n(s_{(2^{n+1}-2,2j+1)} + s_{(2^{n+1}-1+2j)}) &= s_{(2^{n+1}-1+2(2^n-1),2j+1)}, \text{ for } 1 \leq j \leq 2^n - 2; \\ Q_n(s_{(2^{n+1}+2i,2^{n+1}-2)}) &= s_{(2^{n+1}-1+2(2^n-1),2^{n+1}+2i)}, \text{ for } 0 \leq i \leq 2^n - 3; \\ Q_n(s_{(2^{n+1}-2+2(2^n-1),2j-1)}) &= s_{(2^{n+1}-1+2(2^n-1),2^{n+1}-3+2j)}, \text{ for } 1 \leq j \leq 2^n - 1; \\ Q_n(s_{(2^{n+1}-2+2(2^n-1),2^{n+1}-2)}) &= s_{(2^{n+1}-1+2(2^n-1),2^{n+1}-2+2(2^n-1))}; \end{aligned}$$

and Q_n sends all other elements of $\mathcal{B}_{2^{n+1}+2(2^n-1)}$ to zero. Therefore,

$$\begin{aligned} \delta([s_{(2^{n+1}-2+2(2^n-1),2j-1)}]) &= [s_{(2^{n+1}-1+2(2^n-1),2^{n+1}-3+2j)}] = [Q_n(s_{(2^{n+1}-1+2(2^n-1),2j-2)})], \text{ for } 1 \leq j \leq 2^n - 1; \\ \delta([s_{(2^{n+1}-2,1)} + s_{(2^{n+1}-1)}]) &= [s_{(2^{n+1}-1+2(2^n-1),1)}]; \\ \delta([s_{(2^{n+1}-2,2j+1)} + s_{(2^{n+1}-1+2j)}]) &= [s_{(2^{n+1}-1+2(2^n-1),2j+1)}], \text{ for } 1 \leq j \leq 2^n - 2; \\ \delta([s_{(2^{n+1}+2i,2^{n+1}-2)}]) &= [s_{(2^{n+1}-1+2(2^n-1),2^{n+1}+2i)}], \text{ for } 0 \leq i \leq 2^n - 3; \\ \delta([s_{(2^{n+1}-2+2(2^n-1),2^{n+1}-2)}]) &= [s_{(2^{n+1}-1+2(2^n-1),2^{n+1}-2+2(2^n-1))}]; \end{aligned}$$

and δ sends all other elements to zero. Thus, δ is an epimorphism and the set

$$\begin{aligned} \mathcal{B}_{2^{n+1}-1+2 \cdot 2^n} &= \{s_{(2j,2j)} \mid 0 \leq j \leq 2^n - 1\} \\ &\cup \{s_{(2^{n+1}-2+2(i+j),2i+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid 1 \leq i+j \leq 2^n - 2\} \\ &\cup \{s_{(2^{n+1}-2+2(2^n-1),2j-1)} + s_{(2^{n+1}-1+2(2^n-1),2j-2)} \mid 1 \leq j \leq 2^n - 1\} \\ &\cup \{s_{(2^{n+1}-2+2i,2(i+j)+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid 2 \leq i+j \leq 2^n - 2; j \geq 1; i \geq 1\} \\ &\cup \{s_{(2^{n+1}-3+2(i-j),2i+1)} + s_{(2^{n+1}-2+2(i-j),2i)} \mid 0 \leq i \leq 2^n - 2; i \leq j \leq 2^n - 2\} \\ &\cup \{s_{(2^{n+1}+2i,2^{n+1}+2i-2j)} \mid 0 \leq i \leq 2^n - 3; 0 \leq j \leq i\} \cup \{s_{(2^{n+1}-2+2(2^n-1),2^{n+1}-2+2(2^n-1)-2j)} \mid 0 \leq j \leq 2^n - 2\} \end{aligned}$$

represents an $\mathbb{F}_2$ -basis of the kernel of δ and hence an $\mathbb{F}_2$ -basis of $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2 \cdot 2^n}); \mathbb{F}_2)$.

Hence, part (2) holds for $s = 0$.

Suppose that the set

$$\begin{aligned} \mathcal{B}_{2^{n+1}-1+2(2^n+s)} &= \{s_{(2j,2j)} \mid 0 \leq j \leq 2^n + s - 1\} \\ &\cup \{s_{(2^{n+1}-2+2(i+j),2i+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid s+1 \leq i+j \leq 2^n - 2\} \\ &\cup \{s_{(2^{n+1}-2+2i,2j-1)} + s_{(2^{n+1}-1+2i,2j-2)} \mid 2^n - 1 \leq i \leq 2^n - 1 + s; 1 \leq j \leq 2^n - 1\} \\ &\cup \{s_{(2^{n+1}-2+2i,2(i+j)+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid i+j \leq 2^n - 2; j \geq 1; i \geq s+1\} \\ &\cup \{s_{(2^{n+1}-3+2(i-j),2i+1)} + s_{(2^{n+1}-2+2(i-j),2i)} \mid 0 \leq i \leq 2^n - 2; i \leq j \leq 2^n - 2\} \\ &\cup \{s_{(2^{n+1}-2+2i,2^{n+1}-2+2i-2j)} \mid s+1 \leq i \leq 2^n - 1 + s; 0 \leq j \leq i - (s+1)\} \end{aligned}$$

represents an $\mathbb{F}_2$ -basis of $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2(2^n+s)}); Q_n)$ for $0 \leq s \leq 2^n - 3$. By part (3), $H^*(Gr_2(\mathbb{R}^{2^{n+1}+2(2^n+s)}); Q_n)$ has an $\mathbb{F}_2$ -basis represented by

$$\mathcal{B}_{2^{n+1}+2(2^n+s)} = \mathcal{B}_{2^{n+1}-1+2(2^n+s)} \cup \mathcal{C}_{2^{n+1}+2(2^n+s)}.$$

In $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2(2^n+s+1)}); \mathbb{F}_2)$, we have

$$Q_n(s_{(2^{n+1}-2+2(s+1), 2i+1)} + s_{(2^{n+1}-1+2(s+1), 2i)}) = s_{(2^{n+1}-1+2(2^n+s), 2i+1)}, \text{ for } 0 \leq i \leq s+1;$$

$$Q_n(s_{(2^{n+1}+2s, 2(s+j)+3)} + s_{(2^{n+1}+1+2(s+j), 2s+2)}) = s_{(2^{n+1}-1+2(2^n+s), 2(j+s)+3)}, \text{ for } 1 \leq j \leq 2^n - 3 - s;$$

$$Q_n(s_{(2^{n+1}-2+2i, 2^{n+1}+2s)}) = s_{(2^{n+1}-1+2(2^n+s), 2^{n+1}-2+2i)}, \text{ for } s+2 \leq i \leq 2^n - 1 + s;$$

$$Q_n(s_{(2^{n+1}-2+2(2^n+s), 2j-1)}) = s_{(2^{n+1}-1+2(2^n+s), 2^{n+1}-3+2j)}, \text{ for } 1 \leq j \leq 2^n - 1;$$

$$Q_n(s_{(2^{n+1}-2+2(2^n+s), 2(2^n+s))}) = s_{(2^{n+1}-1+2(2^n+s), 2^{n+1}-2+2(2^n+s))};$$

and Q_n sends all other elements of $\mathcal{B}_{2^{n+1}+2(2^n+s)}$ to zero. Therefore,

$$\delta([s_{(2^{n+1}-2+2(2^n+s), 2j-1)}]) = [s_{(2^{n+1}-1+2(2^n+s), 2^{n+1}-3+2j)}] = [Q_n(s_{(2^{n+1}-1+2(2^n+s), 2j-2)})], \text{ for } 1 \leq j \leq 2^n - 1;$$

$$\delta([s_{(2^{n+1}-2+2(s+1), 2i+1)} + s_{(2^{n+1}-1+2(s+1), 2i)}]) = [s_{(2^{n+1}-1+2(2^n+s), 2i+1)}], \text{ for } 0 \leq i \leq s+1;$$

$$\delta([s_{(2^{n+1}+2s, 2(s+j)+3)} + s_{(2^{n+1}+1+2(s+j), 2s+2)}]) = [s_{(2^{n+1}-1+2(2^n+s), 2(j+s)+3)}], \text{ for } 1 \leq j \leq 2^n - 3 - s;$$

$$\delta([s_{(2^{n+1}-2+2i, 2^{n+1}+2s)}]) = [s_{(2^{n+1}-1+2(2^n+s), 2^{n+1}-2+2i)}], \text{ for } s+2 \leq i \leq 2^n - 1 + s;$$

$$\delta([s_{(2^{n+1}-2+2(2^n+s), 2(2^n+s))}]) = [s_{(2^{n+1}-1+2(2^n+s), 2^{n+1}-2+2(2^n+s))}];$$

and δ sends all other elements to zero. Thus, the set

$$\begin{aligned} \mathcal{B}_{2^{n+1}-1+2(2^n+s+1)} &= \{s_{(2j, 2j)} \mid 0 \leq j \leq 2^n + s\} \\ &\cup \{s_{(2^{n+1}-2+2(i+j), 2i+1)} + s_{(2^{n+1}-1+2(i+j), 2i)} \mid s+2 \leq i+j \leq 2^n - 2\} \\ &\cup \{s_{(2^{n+1}-2+2i, 2j-1)} + s_{(2^{n+1}-1+2i, 2j-2)} \mid 2^n - 1 \leq i \leq 2^n + s; 1 \leq j \leq 2^n - 1\} \\ &\cup \{s_{(2^{n+1}-2+2i, 2(i+j)+1)} + s_{(2^{n+1}-1+2(i+j), 2i)} \mid i+j \leq 2^n - 2; j \geq 1; i \geq s+2\} \\ &\cup \{s_{(2^{n+1}-3+2(i-j), 2i+1)} + s_{(2^{n+1}-2+2(i-j), 2i)} \mid 0 \leq i \leq 2^n - 2; i \leq j \leq 2^n - 2\} \\ &\cup \{s_{(2^{n+1}-2+2i, 2^{n+1}-2+2i-2j)} \mid s+2 \leq i \leq 2^n + s; 0 \leq j \leq i - (s+2)\} \end{aligned}$$

represents an $\mathbb{F}_2$ -basis of the kernel of δ , and hence an $\mathbb{F}_2$ -basis of $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2(2^n+s+1)}); Q_n)$. Hence, part (2) holds for $0 \leq s \leq 2^n - 2$.

The final case, $\ell = 2^n + s$ for $s \geq 2^n - 1$, follows from the same argument. $\square$

Corollary 3.3. (1) For $2+k \leq 2^{n+1}$,

$$K(n)^*(Gr_2(\mathbb{R}^{2+k})) \cong K(n)^* \otimes H^*Gr_2(\mathbb{R}^{2+k}).$$

(2) For $2+k > 2^{n+1}$,

$$K(n)^*(Gr_2(\mathbb{R}^{2+k})) \cong K(n)^* \otimes H^*(Gr_2(\mathbb{R}^{2+k}); Q_n).$$

❖ **Conflict of Interest:** Author have no conflict of interest to declare.

❖ **Acknowledgments:** This work is part of the research project CS.2025.B3.046 funded by Saigon University.

REFERENCES

- Hiller, H. (1982). *Geometry of Coxeter groups*. Pitman (Advanced Publishing Program).
- Kuhn, N. J., & Lloyd, C. J. R. (2024). Computing the Morava K -theory of real Grassmannians using chromatic fixed point theory. *Algebraic & Geometric Topology*, 24(2), 919–950. <https://doi.org/10.2140/agt.2024.24.919>
- Lenart, C. (1998). The combinatorics of Steenrod operations on the cohomology of Grassmannians. *Advances in Mathematics*, 136(2), 251–283. <https://doi.org/10.1006/aima.1998.1731>
- Macdonald, I. G. (1995). *Symmetric functions and Hall polynomials* (2nd ed.). Clarendon Press.
- Milnor, J. W., & Stasheff, J. D. (1974). *Characteristic classes*. Princeton University Press.
- Phan, H. C., & Nguyen, D. H. H. (2026). On the Morava K -theory of the real Grassmannian $Gr_2(\mathbb{R}^m)$. *Topology and its Applications*, 377, 109665. <https://doi.org/10.1016/j.topol.2025.109665>
- Yagita, N. (1980). On the Steenrod algebra of Morava K -theory. *Journal of the London Mathematical Society*, 22(3), 423–438. <https://doi.org/10.1112/jlms/s2-22.3.423>

VỀ K -LÍ THUYẾT MORAVA CỦA ĐA TẬP GRASSMANN THỰC $Gr_2(\mathbb{R}^m)$

Phan Hoàng Chơn

Trường Đại học Sài Gòn, Việt Nam

Tác giả liên hệ: Phan Hoàng Chơn – Email: phchon@sgu.edu.vn

Ngày nhận bài: 03-10-2025; Ngày nhận bài sửa: 03-11-2025; Ngày duyệt đăng: 01-12-2025

TÓM TẮT

Mục đích chính của bài báo này là xác định một $K(n)^*$ -cơ sở tường minh của K -lý thuyết Morava thứ n của đa tập Grassmann thực $Gr_2(\mathbb{R}^m)$ với các số nguyên $n \geq 0$ và $m \geq 2$. Sử dụng công thức tổ hợp của Lenart, chúng tôi tính tác động của toán tử Milnor nguyên thủy Q_n lên các hàm đối xứng Schur để xác định Q_n -đồng điều của đối đồng điều modulo 2 của $Gr_2(\mathbb{R}^m)$, từ đó xác định trang $E_{2^{n+1}}$ của dãy phổ Atiyah-Hirzebruch. Chúng tôi chứng minh được số chiều của trang $E_{2^{n+1}}$ bằng với chặn dưới của số chiều của $K(n)^*(Gr_2(\mathbb{R}^m))$ được xác định bởi Kuhn và Lloyd (2024), từ đó nhận được dãy phổ suy biến tại trang $E_{2^{n+1}}$. Kết quả của bài báo cung cấp dữ liệu nền tảng cho việc nghiên cứu tiếp theo cho đa tập Grassmann thực tổng quát.

Từ khóa: dãy phổ Atiyah-Hirzebruch; lý thuyết Morava K ; đa tập Grassmann thực