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Research Article

ON THE MORAVA K-THEORY OF THE REAL GRASSMANNIAN $Gr_2(\mathbb{R}^m)$

Phan Hoang Chon
Saigon University, Vietnam
Corresponding author: Phan Hoang Chon – Email: phchon@sgu.edu.vn
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ABSTRACT

The main purpose of this paper is to determine an explicit $K(n)^*$-basis for the nth Morava K-theory of the real Grassmannian $Gr_2(\mathbb{R}^m)$ for integers $n \geq 0$ and $m \geq 2$. Using Lenart's combinatorial formula, we compute the action of the primitive Milnor operation Q_n on Schur functions to determine the Q_n-homology of the mod 2 cohomology of $Gr_2(\mathbb{R}^m)$, thereby determining the $E_{2^{n+1}}$-term of the Atiyah-Hirzebruch spectral sequence. We show that the size of the $E_{2^{n+1}}$-term equals the lower bound for the size of $K(n)^*(Gr_2(\mathbb{R}^m))$ defined by Kuhn and Lloyd (2024); therefore, the spectral sequence collapses at the $E_{2^{n+1}}$-term. This result provides foundational data for further investigations of Morava K-theory of real Grassmannians.

Keywords: Atiyah-Hirzebruch spectral sequence; Morava K-theory; Real Grassmannian
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1. Introduction

Given a finite complex X, let $K(n)^*(X)$ denote the 2-local nth Morava K-theory of X. This is a graded vector space over the graded field $K(n)_* = \mathbb{F}_2[v_n^{\pm 1}]$ with $|v_n| = 2(2^n - 1)$. For $0 \leq d \leq m$, let $Gr_d(\mathbb{R}^m)$ be the real Grassmannian, namely the manifold of d-planes of $\mathbb{R}^m$. The purpose of this paper is to determine an explicit $K(n)^*$-basis of $K(n)^*(Gr_2(\mathbb{R}^m))$ using the Atiyah–Hirzebruch spectral sequence (AHSS).

For a finite complex X, the AHSS that computes $K(n)^*(X)$ is given by

$$E_2^{s,t}(X) = H^s(X, K(n)^t) \Rightarrow K(n)^{s+t}(X),$$

where the differential $d_r : E_r^{s,t}(X) \to E_r^{s+r,t-r+1}(X)$. This is a spectral sequence of graded $K(n)^* = \mathbb{F}_2[v_n^{\pm 1}]$-algebras, where v_n has cohomological degree $2 - 2^{n+1}$. The sparseness of

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the rows shows that the differential d_r vanishes unless $r = s(2^{n+1}-2)+1$ for some s. Yagita (1980) showed that the first nontrivial differential in the spectral sequence is $d_{2^{n+1}-1}$; moreover, for $x \in E_2^{*,0}(X) \cong H^*(X)$,

$$d_{2^{n+1}-1}(x) = Q_n(x) \otimes v_n.$$

Here, Q_n is the primitive Milnor operation, defined by

$$Q_0 = Sq^1, \quad Q_n = Q_{n-1}Sq^{2^n} + Sq^{2^n}Q_{n-1}.$$

Therefore, determining the $E_{2^{n+1}}$-term of the AHSS is equivalent to determining the Q_n-homology of $H^*(X)$. Set

$$k_n(X) = \dim_{K(n)_*} K(n)^*(X),$$

the rank of $K(n)^*(X)$ over the graded field $K(n)_*$, and

$$k_{Q_n}(X) = \dim_{\mathbb{F}_2} H_*(H^*(X); Q_n),$$

the $\mathbb{F}_2$-dimension of the Margolis homology of H^*X with respect to the Milnor primitive operation Q_n of the mod 2 Steenrod algebra A:

$$H^*(X; Q_n) := \mathrm{Ker}[Q_n : H^*X \to H^*X] / \mathrm{Im}[Q_n : H^*X \to H^*X].$$

Here and below, H^*X denotes the mod 2 cohomology of X.

Using chromatic fixed-point theory, Kuhn and Lloyd (2024) proved the following result.

Theorem 1.1. (Kuhn and Lloyd (2024)). (1) *If* $m \le 2^{n+1}$, *then* $k_n(Gr_d(\mathbb{R}^m)) = \binom{m}{d}$.

(2) *If* $m = 2^{n+1} - \epsilon + 2\ell$, $\ell \ge 1$, $\epsilon = 0,1$, *then*

$$k_{Q_n}(Gr_d(\mathbb{R}^m)) \ge k_n(Gr_d(\mathbb{R}^m)) \ge \sum_{i=0}^{[d/2]} \binom{2^{n+1}-\epsilon}{d-2i}\binom{\ell}{i}.$$

They also conjectured that the inequalities in Theorem 1.1 are always equalities:

Conjecture 1.2. (Kuhn and Lloyd (2024)). *If* $m = 2^{n+1} - \epsilon + 2\ell$, $\ell \ge 1$, $\epsilon = 0,1$, *then*

$$k_{Q_n}(Gr_d(\mathbb{R}^m)) = k_n(Gr_d(\mathbb{R}^m)) = \sum_{i=0}^{[d/2]} \binom{2^{n+1}-\epsilon}{d-2i}\binom{\ell}{i}.$$

It is straightforward to verify that Conjecture 1.2 holds for $d = 1$. Kuhn and Lloyd (2024) proved that Conjecture 1.2 holds for $d = 2$; however, they did not identify an explicit basis for $K(n)^*(Gr_2(\mathbb{R}^m))$.

In Phan and Nguyen (2026), we verified that Conjecture 1.2 holds for $d = 3$. To treat the case $d > 3$ within our framework, we require an explicit $K(n)^*$-basis of $K(n)^*(Gr_2(\mathbb{R}^m))$ as well as an $\mathbb{F}_2$-basis of Q_n-homology of $H^*(Gr_2(\mathbb{R}^m))$. This need motivates the present work. Although the size of $K(n)^*(Gr_2(\mathbb{R}^m))$ is known (see Kuhn & Lloyd, 2024), the

problem of determining a $K(n)^*$-basis of $K(n)^*(Gr_2(\mathbb{R}^m))$ remains nontrivial. To solve this problem, we compute the kernel of the connecting homomorphism δ in the long exact sequence (2.4) (see below) and determine an $\mathbb{F}_2$-basis of $H^*(Gr_2(\mathbb{R}^m);Q_n)$ when m is odd.

The paper is organized into three sections. Section 2 reviews the cohomology of real Grassmannians, Lenart's formula, and a short exact sequence needed for the proofs. Section 3 presents the main results.

2. Preliminaries

To fix notation, we first recall the description of the mod 2 cohomology of the real Grassmannian as a quotient of the ring of symmetric polynomials.

Let $Gr_d(\mathbb{R}^{d+k})$ be the real Grassmann manifold of d-planes in $\mathbb{R}^{d+k}$. The natural filtration

$$Gr_d(\mathbb{R}^d) \subset Gr_d(\mathbb{R}^{d+1}) \subset \cdots \subset Gr_d(\mathbb{R}^{d+k}) \subset \cdots$$

defines the infinite real Grassmann manifold as a colimit $Gr_d(\mathbb{R}^\infty) := \operatorname{colim}_k Gr_d(\mathbb{R}^{d+k})$.

Let γ_1 be the canonical line bundle over the infinite real projective space $\mathbb{R}P^\infty$, and let $\gamma_1^{\times d}$ be the d-plane bundle over $(\mathbb{R}P^\infty)^d$. The bundle $\gamma_1^{\times d}$ is classified by a map $g:(\mathbb{R}P^\infty)^d \to Gr_d(\mathbb{R}^\infty)$. This map induces a monomorphism

$$g^*: H^*Gr_d(\mathbb{R}^\infty) \to H^*(\mathbb{R}P^\infty)^d$$

whose image is the subring of symmetric polynomials in $H^*(\mathbb{R}P^\infty)^d \cong \mathbb{F}_2[x_1,\ldots,x_d]$. Moreover, g^* sends the i-th Stiefel-Whitney class $w_i = w_i(\gamma_d)$ of the canonical d-plane bundle over $Gr_d(\mathbb{R}^\infty)$ to the i-th elementary symmetric polynomial e_i in $x_1,\ldots,x_d$. The mod 2 cohomology of $Gr_d(\mathbb{R}^\infty)$ is therefore the polynomial algebra described by Milnor and Stasheff (1974):

$$H^*Gr_d(\mathbb{R}^\infty) \cong \mathbb{F}_2[w_1,\ldots,w_d].$$

The dual classes $\bar{w}_1, \bar{w}_2, \ldots$ are defined by the equation

$$(1+w_1+\cdots+w_d)(1+\bar{w}_1+\bar{w}_2+\cdots)=1, \quad (2.1)$$

and this allows one to write each $\bar{w}_k$ as a polynomial in $w_1,\ldots,w_d$. The class $\bar{w}_k$ can be interpreted as $w_k(\gamma_d^\perp)$ where $\gamma_d^\perp$ is the k-dimensional bundle complementary to γ_d. For dimensional reasons, in $H^*Gr_d(\mathbb{R}^{d+k})$, $\bar{w}_j = 0$ for all $j > k$. Moreover, the inclusion $Gr_d(\mathbb{R}^{d+k}) \to Gr_d(\mathbb{R}^\infty)$ induces a ring epimorphism

$$H^*Gr_d(\mathbb{R}^\infty) \twoheadrightarrow H^*Gr_d(\mathbb{R}^{d+k})$$

whose kernel is given by the ideal $J(d,k) = (\bar{w}_j \mid j > k)$.

We recall the cell structure of $Gr_d(\mathbb{R}^{d+k})$ as described in Milnor and Stasheff (1974) (see also Hiller (1982)). A Schubert symbol $\lambda = (\lambda_1,\ldots,\lambda_d)$ of $Gr_d(\mathbb{R}^{d+k})$ is a sequence of integers

$$k \geq \lambda_1 \geq \cdots \geq \lambda_d \geq 0.$$

The weight of λ is denoted by $|\lambda|$ and is defined by

$$|\lambda| = \lambda_1 + \cdots + \lambda_d.$$

Thus, a Schubert symbol λ is a partition contained in a $d \times k$ grid when depicted as a Young diagram.

Each Schubert symbol λ determines a Schubert cell Ω_λ of dimension $|\lambda|$ in $Gr_d(\mathbb{R}^{d+k})$, defined by

$$\Omega_\lambda = \{V \in Gr_d(\mathbb{R}^{d+k}) : \dim(V \cap \mathbb{R}^{i+\lambda_{d+1-i}}) = i, 1 \leq i \leq d+k\}.$$

The duals of the Schubert cells form a $\mathbb{Z}_2$-basis of $H^*Gr_d(\mathbb{R}^{d+k})$. Moreover, the dual of Ω_λ, denoted by $s_\lambda \in H^{|\lambda|}Gr_d(\mathbb{R}^{d+k})$, corresponds to the Schur symmetric function associated with the partition λ (see Macdonald (1995)). The class w_i in the Borel presentation corresponds to the elementary symmetric function $s_{(1^i)} = e_i$ and the dual class $\overline{w}_j$ to $s_{(j)} = h_j$, the j-th complete symmetric polynomial. The ring structure of $H^*Gr_d(\mathbb{R}^{d+k})$ in this basis is described by the Littlewood-Richardson rule from the theory of symmetric functions.

Recall that the Milnor operation Q_n in the Steenrod algebra A is recursively defined by

$$Q_0 = Sq^1; \quad Q_{n+1} = Q_n Sq^{2^n} + Sq^{2^n} Q_n, \quad n \geq 0.$$

To state Lenart's (1998) formula for the action of Q_n on Schur symmetric functions, we need some combinatorial definitions.

Given a Young diagram λ contained in another Young diagram μ, one can form the complement of λ in μ, denoted μ / λ and called a skew diagram. The content of a box b of a Young diagram μ in row i and column j is defined by $c(b) = j - i$. For a box b in the skew diagram μ / λ, its content is the content of b embedded in μ.

A skew shape is connected if any two boxes in the diagram can be joined by a sequence of edge-adjacent boxes. A skew shape that does not contain a 2×2-block of boxes is called a broken border strip. A connected broken border strip is called a border strip. If λ is a broken border strip, $cc(\lambda)$ denotes the number of connected components of λ. A sharp corner of a broken border strip is a box with no northern, western, or northwestern neighbor. A dull corner of a broken strip is a box with both northern and western neighbors but no northwestern neighbor. Let $C(\mu / \lambda)$ denote the set of sharp and dull corners of μ / λ.

We are now ready to state the formula in Lenart (1998, Proposition 5.1) for the action of the Milnor operation on a Schubert basis class:

Proposition 2.1. *We have*

$$Q_n(s_\lambda) = \sum_{\mu \supset \lambda : |\mu| - |\lambda| = 2^{n+1}-1; cc(\mu/\lambda) \leq 2} d_{\lambda\mu} s_\mu, \qquad (2.2)$$

where μ / λ *must be a broken border strip and*

$$d_{\lambda\mu} = \begin{cases} \sum\limits_{b\in C(\mu/\lambda)} c(b), & \text{if } \mu/\lambda \text{ is connected}, \\ 1, & \text{if } \mu/\lambda \text{ is disconnected}. \end{cases} \tag{2.3}$$

Recall that the Schubert basis of $H^*Gr_d(\mathbb{R}^m)$ consists of the classes s_λ satisfying $m-d \geq \lambda_1 \geq \cdots \geq \lambda_d \geq 0$. Let $i: Gr_d(\mathbb{R}^{m-1}) \to Gr_d(\mathbb{R}^m)$ denote the map induced by the natural inclusion $\mathbb{R}^{m-1} \to \mathbb{R}^m$. In cohomology, this map induces a surjection

$$i^*: H^*Gr_d(\mathbb{R}^m) \twoheadrightarrow H^*Gr_d(\mathbb{R}^{m-1})$$

which, in the Schubert basis, is given by

$$i^*(s_{(\lambda_1,\ldots,\lambda_d)}) = \begin{cases} 0, & \text{if } \lambda_1 = m-d, \\ s_{(\lambda_1,\ldots,\lambda_d)}, & \text{if } \lambda_1 \leq m-d-1. \end{cases}$$

Proposition 2.2. *There is a short exact sequence of A-modules*

$$0 \to C_{d,m} \to H^*Gr_d(\mathbb{R}^m) \xrightarrow{i^*} H^*Gr_d(\mathbb{R}^{m-1}) \to 0$$

where the kernel $C_{d,m}$ of i^* has a linear basis consisting of the classes $s_{(\lambda_1,\ldots,\lambda_d)}$ with $m-d = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d \geq 0$.

Remark 2.3. Let $C_d(\mathbb{R}^m)$ denote the cofibre of the inclusion $i: Gr_d(\mathbb{R}^{m-1}) \to Gr_d(\mathbb{R}^m)$. Kuhn and Lloyd (2024) showed that this cofibre is homeomorphic to the Thom space $Gr_{d-1}(\mathbb{R}^{m-1})^{\gamma_{d-1}^\perp}$, where $\gamma_{d-1}^\perp$ is the bundle complementary to the canonical bundle γ_{d-1} over $Gr_{d-1}(\mathbb{R}^{m-1})$. The cofibre sequence $Gr_d(\mathbb{R}^{m-1}) \xrightarrow{i} Gr_d(\mathbb{R}^m) \xrightarrow{p} C_d(\mathbb{R}^m)$ thus induces the short exact sequence in Proposition 2.2. In other words, $C_{d,m} \cong \tilde{H}^*C_d(\mathbb{R}^m)$.

The short exact sequence in Proposition 2.2 induces a long exact sequence in Q_n-homology

$$\cdots \to H^*(C_{d,m}; Q_n) \xrightarrow{p^*} H^*(Gr_d(\mathbb{R}^m); Q_n) \xrightarrow{i^*} H^*(Gr_d(\mathbb{R}^{m-1}); Q_n)$$
$$\xrightarrow{\delta} H^{*+2^{n+1}-1}(C_d(\mathbb{R}^m); Q_n) \to \cdots \tag{2.4}$$

3. Main results

In this section, we use the long exact sequence (2.4) to describe an $\mathbb{F}_2$-basis of Q_n-homology of $H^*Gr_2(\mathbb{R}^m)$, and thereby obtain a $K^*(n)$-basis of $K^*(n)(Gr_2(\mathbb{R}^m))$

To determine an $\mathbb{F}_2$-basis of $H^*(Gr_2(\mathbb{R}^{2+k}); Q_n)$, we first need an $\mathbb{F}_2$-basis of $\widetilde{H}^*(C_2(\mathbb{R}^{2+k}); Q_n)$. Kuhn and Lloyd's result gives the following description of $\widetilde{H}^*(C_2(\mathbb{R}^{2+k}); Q_n)$.

Proposition 3.1. (1) *In* $\widetilde{H}^*(C_2(\mathbb{R}^{2+k}); \mathbb{F}_2)$,

$$Q_n(s_{(k,i)}) = \begin{cases} s_{(k,2^{n+1}-1+i)}, & \text{if } i \text{ is even and } k-i \le 2^{n+1}-1, \\ 0, & \text{otherwise.} \end{cases}$$

(2) *If* $2+k \le 2^{n+1}$, *then* Q_n *acts as zero on* $H^*(C_2(\mathbb{R}^{2+k});\mathbb{F}_2)$. *Thus* $\tilde{k}_{Q_n}(C_2(\mathbb{R}^{2+k})) = k+1$.

(3) *If* $2+k > 2^{n+1}$ *and* k *is even, then the classes* $\{s_{(k,2j-1)} : 1 \le j \le 2^n - 1\}$ *and* $\{s_{(k,k-2j)} : 0 \le j \le 2^n - 1\}$ *represent the* Q_n*-homology classes. Thus,* $\tilde{k}_{Q_n}(C_2(\mathbb{R}^{2+k})) = 2^{n+1}-1$.

(4) *If* $2+k > 2^{n+1}$ *and* k *is odd, then the classes* $\{s_{(k,2j-1)} : 1 \le j \le 2^n - 1\}$ *and* $\{s_{(k,k-1-2j)} : 0 \le j \le 2^n - 2\}$ *represent the* Q_n*-homology classes. Thus,* $\tilde{k}_{Q_n}(C_2(\mathbb{R}^{2+k})) = 2^{n+1}-2$.

We now compute a basis of $H^*(Gr_2(\mathbb{R}^{2+k});Q_n)$.

Theorem 3.2. (1) *If* $2+k \le 2^{n+1}$, $H^*(Gr_2(\mathbb{R}^{2+k});Q_n)$ *is isomorphic to* $H^*(Gr_2(\mathbb{R}^{2+k});\mathbb{F}_2)$.

(2) *If* $2+k = 2^{n+1}-1+2\ell$, *then* $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2\ell});Q_n)$ *has an* $\mathbb{F}_2$*-basis represented by* $\mathcal{B}_{2^{n+1}-1+2\ell}$;

(3) *If* $2+k = 2^{n+1}+2\ell$, *then* $H^*(Gr_2(\mathbb{R}^{2^{n+1}+2\ell});Q_n)$ *has an* $\mathbb{F}_2$*-basis represented by*

$$\mathcal{B}_{2^{n+1}+2\ell} = \mathcal{B}_{2^{n+1}-1+2\ell} \cup \mathcal{C}_{2^{n+1}+2\ell},$$

where

$$\mathcal{C}_{2^{n+1}+2\ell} = \{s_{(2^{n+1}-2+2\ell,2j-1)} \mid 1 \le j \le 2^n - 1\} \cup \{s_{(2^{n+1}-2+2\ell,2^{n+1}-2+2\ell-2j} \mid 0 \le j \le 2^n - 1\},$$

and, for $1 \le \ell \le 2^n - 1$,

$$\mathcal{B}_{2^{n+1}-1+2\ell} = \{s_{(i,j)} \mid 2\ell \le j \le i \le 2(2^n-1)\} \cup \{s_{(2j,2j)} \mid 0 \le j \le \ell - 1\}$$
$$\cup \{s_{(2^{n+1}-2+2(i+j),2i+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid 0 \le i+j \le \ell - 1\}$$
$$\cup \{s_{(2^{n+1}-2+2i,2(i+j)+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid 1 \le i+j \le \ell - 1;\ j \ge 1\}$$
$$\cup \{s_{(2j-2,2i+1)} + s_{(2j-1,2i)} \mid 0 \le i \le \ell - 1; \ell + 1 \le j \le 2^n - 1\}$$
$$\cup \{s_{(2^{n+1}-3+2(i-j),2i+1)} + s_{(2^{n+1}-2+2(i-j),2i)} \mid 0 \le i \le \ell - 1; i \le j \le 2^n - 2\}$$
$$\cup \{s_{(2^{n+1}+2i,2j-1)} \mid 0 \le i \le \ell - 2; \ell + 1 \le j \le 2^n - 1\}$$
$$\cup \{s_{(2^{n+1}+2i,2^{n+1}+2i-2j)} \mid 0 \le i \le \ell - 2; 0 \le j \le 2^n - \ell + i\},$$

for $\ell = 2^n + s$ *with* $0 \le s \le 2^n - 2$,

$$\mathcal{B}_{2^{n+1}-1+2(2^n+s)} = \{s_{(2j,2j)} \mid 0 \le j \le 2^n + s - 1\}$$
$$\cup \{s_{(2^{n+1}-2+2(i+j),2i+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid s+1 \le i+j \le 2^n - 2\}$$

$$\cup\{s_{(2^{n+1}-2+2i,2j-1)}+s_{(2^{n+1}-1+2i,2j-2)} \mid 2^n-1\le i\le 2^n-1+s; 1\le j\le 2^n-1\}$$

$$\cup\{s_{(2^{n+1}-2+2i,2(i+j)+1)}+s_{(2^{n+1}-1+2(i+j),2i)} \mid i+j\le 2^n-2;\ j\ge 1; i\ge s+1\}$$

$$\cup\{s_{(2^{n+1}-3+2(i-j),2i+1)}+s_{(2^{n+1}-2+2(i-j),2i)} \mid 0\le i\le 2^n-2; i\le j\le 2^n-2\}$$

$$\cup\{s_{(2^{n+1}-2+2i,2^{n+1}-2+2i-2j)} \mid s+1\le i\le 2^n-1+s; 0\le j\le i-(s+1)\},$$

for $\ell=2^n+s$ *with* $s\ge 2^n-1$

$$\mathcal{B}_{2^{n+1}-1+2(2^n+s)}=\{s_{(2j,2j)} \mid 0\le j\le 2^n+s-1\}$$

$$\cup\{s_{(2^{n+1}-2+2i,2j-1)}+s_{(2^{n+1}-1+2i,2j-2)} \mid s+1\le i\le 2^n-1+s; 1\le j\le 2^n-1\}$$

$$\cup\{s_{(2^{n+1}-3+2(i-j),2i+1)}+s_{(2^{n+1}-2+2(i-j),2i)} \mid 0\le i\le 2^n-2; i\le j\le 2^n-2\}$$

$$\cup\{s_{(2^{n+1}-2+2i,2^{n+1}-2+2i-2j)} \mid s+1\le i\le 2^n-1+s; 0\le j\le i-(s+1)\}.$$

Part (1) of the theorem follows from the fact that, for $d+k\le 2^{n+1}$, Q_n acts as zero on $H^*(Gr_d(\mathbb{R}^{d+k});\mathbb{F}_2)$ (see Kuhn & Lloyd, 2024).

For $d=2$ with $2+k$ even, the connecting homomorphism δ of (2.4) is zero (Kuhn & Lloyd,2024). Therefore, Proposition 3.1 implies the final claim.

We prove part (2) by induction on ℓ. For the inductive step at $\ell+1$, we use the hypothesis in part (3) for ℓ together with Proposition 3.1. We then compute an $\mathbb{F}_2$-basis of the kernel of the connecting homomorphism δ of (2.4), thereby obtaining the required basis. To compute the kernel of δ, we use Lenart's formulas (2.2)-(2.3) to compute the action of Q_n on elements of $H^*(Gr_d(\mathbb{R}^m))$. In general, this computation is complicated because condition (2.3) is difficult to verify. In our case, however, the following observation simplifies the computation.

Let $x=\sum_{\lambda} s_\lambda \in H^*(Gr_d(\mathbb{R}^m))$ be a cycle representing a nontrivial homology class in $H^*(Gr_d(\mathbb{R}^m);Q_n)$. Then x can be regarded as its own preimage in $H^*(Gr_d(\mathbb{R}^{m+1}))$ under i^*. In $H^*(Gr_d(\mathbb{R}^{m+1}))$,

$$Q_n(x)=\sum_{m-d\ge\mu_1\ge\cdots\ge\mu_d} d_\mu s_\mu+\sum_{m+1-d=\nu_1\ge\cdots\ge\nu_d} d_\nu s_\nu,$$

where d_μ and d_ν belong to $\mathbb{F}_2$. Because $Q_n(x)=0$ in $H^*(Gr_d(\mathbb{R}^m))$, the first sum in the formula above vanishes; it therefore remains to compute d_ν in the second sum. This makes the computation in this case much easier than in the general case.

For example, let $x=s_{(2^{n+1}-2+2\ell,2j-1)}\in H^*(Gr_2(\mathbb{R}^{2^{n+1}+2\ell}))$ be a cycle representing a nontrivial homology class in $H^*(Gr_2(\mathbb{R}^{2^{n+1}+2\ell});Q_n)$ for $\ell\ge 1$ and $1\le j\le 2^n-1$. Then x can

be regarded as its own preimage in $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2(\ell+1)}))$ under the map i. Because $Q_n(x)=0$ in $H^*(Gr_2(\mathbb{R}^{2^{n+1}+2\ell}))$, we have the following identity in $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2(\ell+1)}))$:

$$Q_n(x)=d_t s_{(2^{n+1}-3+2(\ell+1),t)},$$

where $d_t\in\mathbb{F}_2$ and $(2^{n+1}-3+2(\ell+1),t)/(2^{n+1}-2+2\ell,2j-1)$ must be a broken border strip containing exactly $2^{n+1}-1$ boxes. We can directly check that

- if $j\geq \ell+2$, then $t>2^{n+1}-3+2(\ell+1)$, which implies $Q_n(x)=0$;
- if $j\leq \ell$, then $(2^{n+1}-3+2(\ell+1),t)/(2^{n+1}-2+2\ell,2j-1)$ is a disconnected broken border strip, which implies $d_t=1$ (see condition (2.3));
- if $j=\ell+1$, then $(2^{n+1}-3+2(\ell+1),t)/(2^{n+1}-2+2\ell,2j-1)$ is a connected broken border strip with two sharp corners of even content and a dull corner of odd content; therefore, $d_t=1$.

Hence,

$$Q_n(x)=0, \ell+2\leq j\leq 2^n-1;$$

$$Q_n(x)=s_{(2^{n+1}-3+2(\ell+1),2^{n+1}-3+2j)}, 1\leq j\leq \ell+1.$$

We use this strategy throughout to compute the image of Q_n in the proof of (2) and therefore omit the repeated details.

Proof of (2). The proof is divided into three parts corresponding to the three cases for ℓ.

Proof of the case $1\leq \ell\leq 2^n-1$**.** From (1), $k_{Q_n}(Gr_2(\mathbb{R}^{2^{n+1}}))=\begin{pmatrix}2^{n+1}\\2\end{pmatrix}$ and the set

$$\mathcal{B}_{2^{n+1}}=\{s_{(i,j)}\mid 0\leq j\leq i\leq 2^{n+1}-2\}$$

represents an $\mathbb{F}_2$-basis of $H^*(Gr_2(\mathbb{R}^{2^{n+1}});Q_n)$. Since $\mathcal{B}_{2^{n+1}}$ can be regarded as its own preimage in $H^*(Gr_2(\mathbb{R}^{2^{n+1}+1}))$ under i, in $H^*(Gr_2(\mathbb{R}^{2^{n+1}+1}))$, we have

$$Q_n(s_{(i,1)})=s_{(2^{n+1}-1,i+1)}, \text{for } 1\leq i\leq 2^{n+1}-2;$$

$$Q_n(s_{(i)})=s_{(2^{n+1}-1,i)}, \text{for } 1\leq i\leq 2^{n+1}-2;$$

and Q_n sends all other elements of $\mathcal{B}_{2^{n+1}}$ to zero. Therefore, since $\delta([x])=[Q_n(i^{-1}(x))]$,

$$\delta([s_{(2j-2,1)}])=s_{(2^{n+1}-1,2j-1)}, \text{for } 2\leq j\leq 2^n-1;$$

$$\delta([s_{(2^{n+1}-2,1)}])=[Q_n(s_{(2^{n+1}-1)})];$$

$$\delta([s_{(2^{n+1}-3-2j,1)}])=[s_{(2^{n+1}-1,2^{n+1}-2-2j)}], \text{for } 0\leq j\leq 2^n-2;$$

$$\delta([s_{(2j-1)}])=[s_{(2^{n+1}-1,2j-1)}], \text{for } 1\leq j\leq 2^n-1;$$

$$\delta([s_{(2^{n+1}-2-2j)}])=[s_{(2^{n+1}-1,2^{n+1}-2-2j)}], \text{for } 0\leq j\leq 2^n-2;$$

and δ sends all other elements to zero. Thus, δ is an epimorphism, and the set

$$\mathcal{B}_{2^{n+1}+1} = \{s_{(i,j)} : 2 \le j \le i \le 2^{n+1}-2\} \cup \{s_{(2j,2j)} \mid 0 \le j \le 0\}$$
$$\cup \{s_{(2^{n+1}-2+2j,1)} + s_{(2^{n+1}-1+2j)} \mid 0 \le j \le 0\}$$
$$\cup \{s_{(2j-2,1)} + s_{(2j-1)} \mid 2 \le j \le 2^n - 1\}$$
$$\cup \{s_{(2^{n+1}-3-2j,1)} + s_{(2^{n+1}-2-2j)} \mid 0 \le j \le 2^n - 2\}$$

represents an $\mathbb{F}_2$-basis of the kernel of δ, and hence of $H^*(Gr_2(\mathbb{R}^{2^{n+1}+1}); Q_n)$. Hence, part (2) holds for $\ell = 1$.

Assume that for $1 \le \ell \le 2^n - 2$, $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2\ell}); Q_n)$ has an $\mathbb{F}_2$-basis represented by

$$\mathcal{B}_{2^{n+1}-1+2\ell} = \{s_{(i,j)} \mid 2\ell \le j \le i \le 2(2^n-1)\} \cup \{s_{(2j,2j)} \mid 0 \le j \le \ell - 1\}$$
$$\cup \{s_{(2^{n+1}-2+2(i+j),2i+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid 0 \le i + j \le \ell - 1\}$$
$$\cup \{s_{(2^{n+1}-2+2i,2(i+j)+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid 1 \le i + j \le \ell - 1;\ j \ge 1\}$$
$$\cup \{s_{(2j-2,2i+1)} + s_{(2j-1,2i)} \mid 0 \le i \le \ell - 1; \ell + 1 \le j \le 2^n - 1\}$$
$$\cup \{s_{(2^{n+1}-3+2(i-j),2i+1)} + s_{(2^{n+1}-2+2(i-j),2i)} \mid 0 \le i \le \ell - 1; i \le j \le 2^n - 2\}$$
$$\cup \{s_{(2^{n+1}+2i,2j-1)} \mid 0 \le i \le \ell - 2; \ell + 1 \le j \le 2^n - 1\}$$
$$\cup \{s_{(2^{n+1}+2i,2^{n+1}+2i-2j)} \mid 0 \le i \le \ell - 2; 0 \le j \le 2^n - \ell + i\},$$

By part (3), $H^*(Gr_2(\mathbb{R}^{2^{n+1}+2\ell}); Q_n)$ has an $\mathbb{F}_2$-basis represented by

$$\mathcal{B}_{2^{n+1}+2\ell} = \mathcal{B}_{2^{n+1}-1+2\ell} \cup \mathcal{C}_{2^{n+1}+2\ell},$$

where

$$\mathcal{C}_{2^{n+1}+2\ell} = \{s_{(2^{n+1}+2(\ell-1),2j-1)} \mid 1 \le j \le 2^n - 1\} \cup \{s_{(2^{n+1}+2(\ell-1),2^{n+1}+2(\ell-1)-2j)} \mid 0 \le j \le 2^n - 1\}.$$

Applying the preceding observation, we obtain, in $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2(\ell+1)}))$,

$$Q_n(s_{(i,2\ell+1)}) = s_{(2^{n+1}-1+2\ell,i+1)}, \text{for } 2\ell + 1 \le i \le 2(2^n - 1);$$
$$Q_n(s_{(i,2\ell)}) = s_{(2^{n+1}-1+2\ell,i)}, \text{for } 2\ell + 1 \le i \le 2(2^n - 1);$$
$$Q_n(s_{(2\ell,2i+1)} + s_{(2\ell+1,2i)}) = s_{(2^{n+1}-1+2\ell,2i+1)}, \text{for } 0 \le i \le \ell - 1;$$
$$Q_n(s_{(2^{n+1}+2i,2\ell+1)}) = s_{(2^{n+1}-1+2\ell,2^{n+1}+2i+1)}, \text{for } 0 \le i \le \ell - 2;$$
$$Q_n(s_{(2^{n+1}+2i,2\ell)}) = s_{(2^{n+1}-1+2\ell,2^{n+1}+2i)}, \text{for } 0 \le i \le \ell - 2;$$
$$Q_n(s_{(2^{n+1}+2(\ell-1),2j-1)}) = s_{(2^{n+1}-1+2\ell,2^{n+1}-3+2j)}, \text{for } 1 \le j \le \ell + 1;$$
$$Q_n(s_{(2^{n+1}+2(\ell-1),2\ell)}) = s_{(2^{n+1}-1+2\ell,2^{n+1}-2+2\ell)};$$

and Q_n sends all other elements of $\mathcal{B}_{2^{n+1}+2\ell}$ to zero. Therefore,

$$\delta([s_{(2^{n+1}+2(\ell-1),2j-1)}]) = [s_{(2^{n+1}-1+2\ell,2^{n+1}-3+2j)}] = [Q_n(s_{(2^{n+1}-1+2\ell,2(j-1))})], \text{for } 1 \le j \le \ell + 1;$$

$$\delta([s_{(2^{n+1}-2,2\ell+1)}]) = [s_{(2^{n+1}-1+2\ell,2^{n+1}-1)}] = [Q_n(s_{(2^{n+1}-1+2\ell)})];$$

$$\delta([s_{(2^{n+1}+2i,2\ell+1)}]) = [s_{(2^{n+1}-1+2\ell,2^{n+1}+2i+1)} = [Q_n(s_{(2^{n+1}-1+2\ell,2(i+1))})], \text{for } 0 \le i \le \ell-2;$$

$$\delta([s_{(2j-2,2\ell+1)}]) = [s_{(2^{n+1}-1+2\ell,2j-1)}], \text{for } \ell+2 \le j \le 2^n-1;$$

$$\delta([s_{(2^{n+1}-3+2(\ell-j),2\ell+1)}]) = [s_{(2^{n+1}-1+2\ell,2^{n+1}-2+2(\ell-j))}], \text{for } \ell \le j \le 2^n-2;$$

$$\delta([s_{(2j-1,2\ell)}]) = [s_{(2^{n+1}-1+2\ell,2j-1)}], \text{for } \ell+1 \le j \le 2^n-1;$$

$$\delta([s_{(2^{n+1}-2+2(\ell-j),2\ell)}]) = [s_{(2^{n+1}-1+2\ell,2^{n+1}-2+2(\ell-j))}], \text{for } \ell \le j \le 2^n-2;$$

$$\delta([s_{(2\ell,2i+1)} + s_{(2\ell+1,2i)}]) = [s_{(2^{n+1}-1+2\ell,2i+1)}], \text{for } 0 \le i \le \ell-1;$$

$$\delta([s_{(2^{n+1}+2i,2\ell)}]) = [s_{(2^{n+1}-1+2\ell,2^{n+1}+2i)}], \text{for } 0 \le i \le \ell-2;$$

$$\delta([s_{(2^{n+1}+2(\ell-1),2\ell)}]) = [s_{(2^{n+1}-1+2\ell,2^{n+1}-2+2\ell)}];$$

and δ sends all other elements to zero. Thus, δ is an epimorphism and the set

$$\mathcal{B}_{2^{n+1}-1+2(\ell+1)} = \{s_{(i,j)} \mid 2(\ell+1) \le j \le i \le 2(2^n-1)\} \cup \{s_{(2j,2j)} \mid 0 \le j \le \ell\}$$

$$\cup \{s_{(2^{n+1}-2+2(i+j),2i+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid 0 \le i+j \le \ell\}$$

$$\cup \{s_{(2^{n+1}-2+2i,2(i+j)+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid 1 \le i+j \le \ell;\ j \ge 1\}$$

$$\cup \{s_{(2j-2,2i+1)} + s_{(2j-1,2i)} \mid 0 \le i \le \ell;\ \ell+2 \le j \le 2^n-1\}$$

$$\cup \{s_{(2^{n+1}-3+2(i-j),2i+1)} + s_{(2^{n+1}-2+2(i-j),2i)} \mid 0 \le i \le \ell;\ i \le j \le 2^n-2\}$$

$$\cup \{s_{(2^{n+1}+2i,2j-1)} \mid 0 \le i \le \ell-1;\ \ell+2 \le j \le 2^n-1\}$$

$$\cup \{s_{(2^{n+1}+2i,2^{n+1}+2i-2j)} \mid 0 \le i \le \ell-1;\ 0 \le j \le 2^n-(\ell+1)+i\}$$

represents an $\mathbb{F}_2$-basis of the kernel of δ, and hence an $\mathbb{F}_2$-basis of $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2(\ell+1)}); Q_n)$.

Thus, part (2) holds for $1 \le \ell \le 2^n-1$.

Proof of the case $\ell \ge 2^n$**.** In this case, set $\ell = 2^n + s$ for $s \ge 0$. By the preceding result for $\ell = 2^n-1$ and part (3), $H^*(Gr_2(\mathbb{R}^{2^{n+1}+2(2^n-1)}); Q_n)$ has an $\mathbb{F}_2$-basis represented by

$$\mathcal{B}_{2^{n+1}+2(2^n-1)} = \{s_{(2j,2j)} \mid 0 \le j \le 2^n-1\}$$

$$\cup \{s_{(2^{n+1}-2+2(i+j),2i+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid 0 \le i+j \le 2^n-2\}$$

$$\cup \{s_{(2^{n+1}-2+2i,2(i+j)+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid 1 \le i+j \le 2^n-2;\ j \ge 1\}$$

$$\cup \{s_{(2^{n+1}-3+2(i-j),2i+1)} + s_{(2^{n+1}-2+2(i-j),2i)} \mid 0 \le i \le 2^n-2; i \le j \le 2^n-2\}$$

$$\cup \{s_{(2^{n+1}+2i,2^{n+1}+2i-2j)} \mid 0 \le i \le 2^n-3; 0 \le j \le i+1\}$$

$$\cup \{s_{(2^{n+1}-2+2(2^n-1),2j-1)} \mid 1 \le j \le 2^n-1\} \cup \{s_{(2^{n+1}-2+2(2^n-1),2^{n+1}-2+2(2^n-1)-2j)} \mid 0 \le j \le 2^n-1\}.$$

For $\ell = 2^n$, in $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2\cdot 2^n}); \mathbb{F}_2)$, we have

$$Q_n(s_{(2^{n+1}-2,1)} + s_{(2^{n+1}-1)}) = s_{(2^{n+1}-1+2(2^n-1),1)};$$

$$Q_n(s_{(2^{n+1}-2,2j+1)} + s_{(2^{n+1}-1+2j)}) = s_{(2^{n+1}-1+2(2^n-1),2j+1)}, \text{for } 1 \le j \le 2^n - 2;$$

$$Q_n(s_{(2^{n+1}+2i,2^{n+1}-2)}) = s_{(2^{n+1}-1+2(2^n-1),2^{n+1}+2i)}, \text{for } 0 \le i \le 2^n - 3;$$

$$Q_n(s_{(2^{n+1}-2+2(2^n-1),2j-1)}) = s_{(2^{n+1}-1+2(2^n-1),2^{n+1}-3+2j)}, \text{for } 1 \le j \le 2^n - 1;$$

$$Q_n(s_{(2^{n+1}-2+2(2^n-1),2^{n+1}-2)}) = s_{(2^{n+1}-1+2(2^n-1),2^{n+1}-2+2(2^n-1))};$$

and Q_n sends all other elements of $\mathcal{B}_{2^{n+1}+2(2^n-1)}$ to zero. Therefore,

$$\delta([s_{(2^{n+1}-2+2(2^n-1),2j-1)}]) = [s_{(2^{n+1}-1+2(2^n-1),2^{n+1}-3+2j)}] = [Q_n(s_{(2^{n+1}-1+2(2^n-1),2j-2)})], \text{for } 1 \le j \le 2^n - 1;$$

$$\delta([s_{(2^{n+1}-2,1)} + s_{(2^{n+1}-1)}]) = [s_{(2^{n+1}-1+2(2^n-1),1)}];$$

$$\delta([s_{(2^{n+1}-2,2j+1)} + s_{(2^{n+1}-1+2j)}]) = [s_{(2^{n+1}-1+2(2^n-1),2j+1)}], \text{for } 1 \le j \le 2^n - 2;$$

$$\delta([s_{(2^{n+1}+2i,2^{n+1}-2)}]) = [s_{(2^{n+1}-1+2(2^n-1),2^{n+1}+2i)}], \text{for } 0 \le i \le 2^n - 3;$$

$$\delta([s_{(2^{n+1}-2+2(2^n-1),2^{n+1}-2)}]) = [s_{(2^{n+1}-1+2(2^n-1),2^{n+1}-2+2(2^n-1))}];$$

and δ sends all other elements to zero. Thus, δ is an epimorphism and the set

$$\begin{aligned}
&\mathcal{B}_{2^{n+1}-1+2\cdot 2^n} = \{s_{(2j,2j)} \mid 0 \le j \le 2^n - 1\} \\
&\qquad \cup \{s_{(2^{n+1}-2+2(i+j),2i+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid 1 \le i + j \le 2^n - 2\} \\
&\cup \{s_{(2^{n+1}-2+2(2^n-1),2j-1)} + s_{(2^{n+1}-1+2(2^n-1),2j-2)} \mid 1 \le j \le 2^n - 1\} \\
&\cup \{s_{(2^{n+1}-2+2i,2(i+j)+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid 2 \le i + j \le 2^n - 2;\ j \ge 1; i \ge 1\} \\
&\cup \{s_{(2^{n+1}-3+2(i-j),2i+1)} + s_{(2^{n+1}-2+2(i-j),2i)} \mid 0 \le i \le 2^n - 2; i \le j \le 2^n - 2\} \\
&\cup \{s_{(2^{n+1}+2i,2^{n+1}+2i-2j)} \mid 0 \le i \le 2^n - 3; 0 \le j \le i\} \cup \{s_{(2^{n+1}-2+2(2^n-1),2^{n+1}-2+2(2^n-1)-2j)} \mid 0 \le j \le 2^n - 2\}
\end{aligned}$$

represents an $\mathbb{F}_2$-basis of the kernel of δ and hence an $\mathbb{F}_2$-basis of $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2\cdot 2^n}); Q_n)$. Hence, part (2) holds for $s = 0$.

Suppose that the set

$$\begin{aligned}
&\mathcal{B}_{2^{n+1}-1+2(2^n+s)} = \{s_{(2j,2j)} \mid 0 \le j \le 2^n + s - 1\} \\
&\cup \{s_{(2^{n+1}-2+2(i+j),2i+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid s + 1 \le i + j \le 2^n - 2\} \\
&\cup \{s_{(2^{n+1}-2+2i,2j-1)} + s_{(2^{n+1}-1+2i,2j-2)} \mid 2^n - 1 \le i \le 2^n - 1 + s; 1 \le j \le 2^n - 1\} \\
&\cup \{s_{(2^{n+1}-2+2i,2(i+j)+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid i + j \le 2^n - 2;\ j \ge 1; i \ge s + 1\} \\
&\cup \{s_{(2^{n+1}-3+2(i-j),2i+1)} + s_{(2^{n+1}-2+2(i-j),2i)} \mid 0 \le i \le 2^n - 2; i \le j \le 2^n - 2\} \\
&\cup \{s_{(2^{n+1}-2+2i,2^{n+1}-2+2i-2j)} \mid s + 1 \le i \le 2^n - 1 + s; 0 \le j \le i - (s+1)\}
\end{aligned}$$

represents an $\mathbb{F}_2$-basis of $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2(2^n+s)}); Q_n)$ for $0 \le s \le 2^n - 3$. By part (3), $H^*(Gr_2(\mathbb{R}^{2^{n+1}+2(2^n+s)}); Q_n)$ has an $\mathbb{F}_2$-basis represented by

$$\mathcal{B}_{2^{n+1}+2(2^n+s)} = \mathcal{B}_{2^{n+1}-1+2(2^n+s)} \cup \mathcal{C}_{2^{n+1}+2(2^n+s)}.$$

In $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2(2^n+s+1)}); \mathbb{F}_2)$, we have

$$Q_n(s_{(2^{n+1}-2+2(s+1),2i+1)} + s_{(2^{n+1}-1+2(s+1),2i)}) = s_{(2^{n+1}-1+2(2^n+s),2i+1)}, \text{for } 0 \le i \le s+1;$$

$$Q_n(s_{(2^{n+1}+2s,2(s+j)+3)} + s_{(2^{n+1}+1+2(s+j),2s+2)}) = s_{(2^{n+1}-1+2(2^n+s),2(j+s)+3)}, \text{for } 1 \le j \le 2^n - 3 - s;$$

$$Q_n(s_{(2^{n+1}-2+2i,2^{n+1}+2s)}) = s_{(2^{n+1}-1+2(2^n+s),2^{n+1}-2+2i)}, \text{for } s+2 \le i \le 2^n - 1 + s;$$

$$Q_n(s_{(2^{n+1}-2+2(2^n+s),2j-1)}) = s_{(2^{n+1}-1+2(2^n+s),2^{n+1}-3+2j)}, \text{for } 1 \le j \le 2^n - 1;$$

$$Q_n(s_{(2^{n+1}-2+2(2^n+s),2(2^n+s)}) = s_{(2^{n+1}-1+2(2^n+s),2^{n+1}-2+2(2^n+s))};$$

and Q_n sends all other elements of $\mathcal{B}_{2^{n+1}+2(2^n+s)}$ to zero. Therefore,

$$\delta([s_{(2^{n+1}-2+2(2^n+s),2j-1)}]) = [s_{(2^{n+1}-1+2(2^n+s),2^{n+1}-3+2j)}] = [Q_n(s_{(2^{n+1}-1+2(2^n+s),2j-2)})], \text{for } 1 \le j \le 2^n - 1;$$

$$\delta([s_{(2^{n+1}-2+2(s+1),2i+1)} + s_{(2^{n+1}-1+2(s+1),2i)}]) = [s_{(2^{n+1}-1+2(2^n+s),2i+1)}], \text{for } 0 \le i \le s+1;$$

$$\delta([s_{(2^{n+1}+2s,2(s+j)+3)} + s_{(2^{n+1}+1+2(s+j),2s+2)}]) = [s_{(2^{n+1}-1+2(2^n+s),2(j+s)+3)}], \text{for } 1 \le j \le 2^n - 3 - s;$$

$$\delta([s_{(2^{n+1}-2+2i,2^{n+1}+2s)}]) = [s_{(2^{n+1}-1+2(2^n+s),2^{n+1}-2+2i)}], \text{for } s+2 \le i \le 2^n - 1 + s;$$

$$\delta([s_{(2^{n+1}-2+2(2^n+s),2(2^n+s)}]) = [s_{(2^{n+1}-1+2(2^n+s),2^{n+1}-2+2(2^n+s))}];$$

and δ sends all other elements to zero. Thus, the set

$$\mathcal{B}_{2^{n+1}-1+2(2^n+s+1)} = \{s_{(2j,2j)} \mid 0 \le j \le 2^n + s\}$$

$$\cup \{s_{(2^{n+1}-2+2(i+j),2i+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid s+2 \le i+j \le 2^n - 2\}$$

$$\cup \{s_{(2^{n+1}-2+2i,2j-1)} + s_{(2^{n+1}-1+2i,2j-2)} \mid 2^n - 1 \le i \le 2^n + s; 1 \le j \le 2^n - 1\}$$

$$\cup \{s_{(2^{n+1}-2+2i,2(i+j)+1)} + s_{(2^{n+1}-1+2(i+j),2i)} \mid i+j \le 2^n - 2;\ j \ge 1; i \ge s+2\}$$

$$\cup \{s_{(2^{n+1}-3+2(i-j),2i+1)} + s_{(2^{n+1}-2+2(i-j),2i)} \mid 0 \le i \le 2^n - 2; i \le j \le 2^n - 2\}$$

$$\cup \{s_{(2^{n+1}-2+2i,2^{n+1}-2+2i-2j)} \mid s+2 \le i \le 2^n + s; 0 \le j \le i - (s+2)\}$$

represents an $\mathbb{F}_2$-basis of the kernel of δ, and hence an $\mathbb{F}_2$-basis of $H^*(Gr_2(\mathbb{R}^{2^{n+1}-1+2(2^n+s+1)}); Q_n)$. Hence, part (2) holds for $0 \le s \le 2^n - 2$.

The final case, $\ell = 2^n + s$ for $s \ge 2^n - 1$, follows from the same argument. □

Corollary 3.3. (1) *For* $2 + k \le 2^{n+1}$,

$$K(n)^*(Gr_2(\mathbb{R}^{2+k})) \cong K(n)^* \otimes H^*Gr_2(\mathbb{R}^{2+k}).$$

(2) *For* $2 + k > 2^{n+1}$,

$$K(n)^*(Gr_2(\mathbb{R}^{2+k})) \cong K(n)^* \otimes H^*(Gr_2(\mathbb{R}^{2+k}); Q_n).$$

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VỀ K-LÍ THUYẾT MORAVA CỦA ĐA TẠP GRASSMANN THỰC $Gr_2(\mathbb{R}^m)$

Phan Hoàng Chơn

Trường Đại học Sài Gòn, Việt Nam

Tác giả liên hệ: Phan Hoàng Chơn – Email: phchon@sgu.edu.vn

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TÓM TẮT

Mục đích chính của bài báo này là xác định một $K(n)^*$-cơ sở tường minh của K-lí thuyết Morava thứ n của đa tạp Grassmann thực $Gr_2(\mathbb{R}^m)$ với các số nguyên $n \geq 0$ và $m \geq 2$. Sử dụng công thức tổ hợp của Lenart, chúng tôi tính tác động của toán tử Milnor nguyên thủy Q_n lên các hàm đối xứng Schur để xác định Q_n-đồng điều của đối đồng điều modulo 2 của $Gr_2(\mathbb{R}^m)$, từ đó xác định trang $E_{2^{n+1}}$ của dãy phổ Atiyah-Hirzebruch. Chúng tôi chứng minh được số chiều của trang $E_{2^{n+1}}$ bằng với chặn dưới của số chiều của $K(n)^*(Gr_2(\mathbb{R}^m))$ được xác định bởi Kuhn và Lloyd (2024), từ đó nhận được dãy phổ suy biến tại trang $E_{2^{n+1}}$. Kết quả của bài báo cung cấp dữ liệu nền tảng cho việc nghiên cứu tiếp theo cho đa tạp Grassmann thực tổng quát.

Từ khóa: dãy phổ Atiyah-Hirzebruch; lí thuyết Morava K; đa tạp Grassmann thực