TRƯỜNG ĐẠI HỌC SƯ PHẠM TP HỒ CHÍ MINH
TẠP CHÍ KHOA HỌC
KHOA HỌC TỰ NHIÊN VÀ CÔNG NGHỆ
Tập 16, Số 3 (2019): 91-102

HO CHI MINH CITY UNIVERSITY OF EDUCATION
JOURNAL OF SCIENCE
NATURAL SCIENCES AND TECHNOLOGY
Vol. 16, No. 3 (2019): 91-102

ISSN: 1859-3100

Email: tapchikhoahoc@hcmue.edu.vn; Website: http://tckh.hcmue.edu.vn

ERROR BOUNDS FOR GENERALIZED MIXED WEAK VECTOR QUASIEQUILIBRIUM PROBLEMS VIA REGULARIZED GAP FUNCTIONS

Vo Minh Tam, Nguyen Huynh Vu Duy, Nguyen Kim Phat
Department of Mathematics Teacher Education – Dong Thap University
Corresponding author: Vo Minh Tam – Email: vmtam@dthu.edu.vn
Received: 13/12/2018; Revised: 01/3/2019; Accepted: 25/3/2019

ABSTRACT

This paper introduces regularized gap functions for a class of generalized mixed weak vector quasiequilibrium problems. Then, error bounds for the concerning problems via regularized gap functions are established. Some examples are provided to illustrate the results.

Keywords: mixed weak vector quasiequilibrium, strong monotonicity, regularized gap function, error bound.

1. Introduction and preliminaries

Throughout in this paper, let $\mathbf{R}^n$ be the n dimensional Euclidean space with the inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let

$$\mathbf{R}^m_+ = \{(y_1,...,y_m)\in\mathbf{R}^m : y_i \ge 0, i=1,2,...,m\}$$

be the nonnegative orthant of $\mathbf{R}^m, A\subset\mathbf{R}^n$ be a nonempty, closed and convex set in $\mathbf{R}^n$. Let $K : A$ Ã A be a set-valued mapping. For each $i\in\{1,2,...,m\}$, let $T_i : A\to\mathbf{R}$ be a continuous function, $\phi_i : A\times A\to\mathbf{R}$, $\eta : A\times A\to\mathbf{R}$ and $F_i : A\times A\to\mathbf{R}$ be continuous bifunctions such that $\eta(x,y)+\eta(y,x)=0$ and $F_i(x,x)=0$ for all $x,y\in A$. Let $F := (F_1,F_2,...,F_m)$, $T := (T_1,T_2,...,T_m)$, $\phi := (\phi_1,\phi_2,...,\phi_m)$ and for any $x,v\in\mathbf{R}$,

$$\langle T(x),v\rangle := (\langle T_1(x),v\rangle,\langle T_2(x),v\rangle,...,\langle T_m(x),v\rangle).$$

In this paper, the authors consider the following *generalized mixed weak vector quasi-equilibrium problem* (shortly, (GMWQEP)) which consists in finding $x\in K(x)$ such that

$$F(x,y)+\langle T(x),\eta(y,x)\rangle+\phi(x,y)-\phi(x,x)\notin -\operatorname{int}\mathbf{R}^m_+, \forall y\in K(x).$$

If $m=1$ then (GMWQEP) reduces to the following *generalized mixed weak quasi-equilibrium problem* (shortly, (GMWQEP)^1) of finding $x\in K(x)$ such that

$$F_1(x,y)+\langle T_1(x),\eta(y,x)\rangle+\phi_1(x,y)-\phi_1(x,x)\ge 0, \forall y\in K(x).$$

The solution sets of problems (GMWQEP) and (GMWQEP)^1 are denoted by S and S^1, respectively. To illustrate motivations for this setting, some special cases of the problem (GMWQEP) are provided.

(a) If $K(x) \equiv A,\ \forall x \in A$ then (GMWQEP) reduces to the following *generalized extended mixed vector equilibrium problem* (shortly, (GEMVEP)) considered by Husain & Singh (2017) of finding $x \in A$ such that

$$F(x,y) + \langle T(x), \eta(y,x)\rangle + \phi(x,y) - \phi(x,x) \notin -\operatorname{int}\mathbf{R}_+^m, \forall y \in A.$$

(b) If $K(x) \equiv A$ and $\eta(y,x) = y - x,\ \forall x, y \in A$, then (GMWQEP) reduces to the following *generalized mixed vector equilibrium problem* (shortly, (GMVEP)) considered by Khan & Chen (2015) of finding $x \in A$ such that

$$F(x,y) + \langle T(x), y - x\rangle + \phi(x,y) - \phi(x,x) \notin -\operatorname{int}\mathbf{R}_+^m, \forall y \in A.$$

(c) If $m = 1,\ K(x) \equiv A,\ F_1 \equiv 0,\ \phi_1 \equiv 0$ and $\eta(y,x) = y - x,\ \forall x, y \in A$, then (GMWQEP) reduces to the following *variational inequality problem* (shortly, (VIP)) studied by Yamashita & Fukushima (1997) of finding $x \in A$ such that

$$\langle T_1(x), y - x\rangle \ge 0, \forall y \in A.$$

(d) If $m = 1,\ \eta \equiv 0$ and $\phi_1 \equiv 0$, then (GMWQEP) reduces to the following *abstract quasiequilibrium problem* (shortly, (QEP)) studied by Bigi & Passacantando (2016) of finding $x \in K(x)$ such that

$$F_1(x,y) \ge 0, \forall y \in K(x).$$

Error bounds which explore the upper estimation of the distance between an arbitrary feasible point and the solution set play an important role in algorithms design for classes of related-optimization problems. The regularized gap function which is an efficient method to investigate error bounds was first introduced by Fukushima (1992) for the variational inequalities. Motivated by Fukushima (1992), based on strong monotonicity assumptions Yamashita & Fukushima (1997) studied global error bounds for general variational inequalities under using regularized gap functions of the Moreau-Yosida type. Since then, the study of error bounds for related-optimization problems has become an interesting and important topic in optimization theory (see Husain & Singh (2017), Khan & Chen (2015), Yamashita & Fukushima (1997), Bigi & Passacantando (2016), Fukushima (1992), Anh, Hung, & Tam (2018), Mastroeni (2003) and the references therein). In Khan & Chen (2015), the regularized gap functions of Fukushima type versions and error bounds were studied for generalized mixed vector equilibrium problems infinite-dimensional spaces. Afer that, Husain & Sighn (2017) extended and improved the main results in Khan & Chen (2015) for the generalized extended mixed vector equilibrium problem. Bigi & Passacantando (2016) investigated some smoothness properties of the gap functions and error bounds for the quasiequilibrium problems. Very recently, Anh et al. (2018) studied regularized gap functions of Fukushia type and Moreau-Yosida type and error bounds for generalized mixed strong vector quasiequilibrium problems in infinite dimensional spaces.

To the best of our knowledge, up to now, there does not exist any work concerning the regularized gap functions of Fukushia type and Moreau-Yosida type and error bounds for (GMWQEP). Therefore, it is interesting to investigate the the regularized gap functions of Fukushia type and Moreau-Yosida type for (GMWQEP). The second aim of this paper is to establish the error bounds for (GMWQEP) by using these regularized gap functions.

Now, some definitions shall be recalled, which will be used in the sequel.

Definition 1.1. (See Rockafellar & Wets (1998))

A real function $F: A \to \mathbf{R}$ is said to be *convex* if for each $x, y \in A$ and $\lambda \in [0,1]$,

$F(\lambda x + (1-\lambda)y) \le \lambda F(x) + (1-\lambda)F(y).$

Definition 1.2. (See Husain & Singh (2017), Khan & Chen (2015))

Let $T: A \to \mathbf{R}$, $\phi: A\times A \to \mathbf{R}$, $F: A\times A \to \mathbf{R}$, $\eta: A\times A \to \mathbf{R}$ be real functions. Then

(i) F is said to be *strongly monotone* with modulus $\alpha > 0$ if, for each $(x, y) \in A\times A$,

$F(x, y) + F(y, x) + \alpha \|x - y\|^2 \le 0;$

(ii) T is said to be η*-strongly monotone* with modulus $\mu > 0$ if,

$\langle T(y) - T(x), \eta(y, x)\rangle - \mu \|x - y\|^2 \ge 0, \ \forall (x, y) \in A\times A;$

(iii) ϕ is said to be *skew-symmetric* if, for each $(x, y) \in A\times A$,

$\phi(x,x) - \phi(x, y) - \phi(y, x) + \phi(y, y) \ge 0.$

Definition 1.3. (See Aubin & Ekeland (1984), Chapter 3, section 1)

Let X and Y be two Hausdorff topological spaces. A set-valued mapping $G: X \rightrightarrows Y$ is said to be

(i) *lower semicontinuous* at $x_0 \in X$, if $G(x_0) \cap U \ne \emptyset$ for some open subset $U \subset Y$ implies the existence of a neighborhood V of x_0 such that $G(x) \cap U \ne \emptyset$ for $x \in V$;

(ii) *upper semicontinuous* at $x_0 \in X$, if for each open neighborhood U of $G(x_0)$, there is a neighborhood V of x_0 such that $U \supset G(x)$ for all $x \in V$.

It is said that G is *lower* (*upper*) *semicontinuous* on a subset A of X if it is lower (upper, respectively) semicontinuous at each $x \in A$. G is said to be *continuous* on A if it is both lower and upper semicontinuous on A. If $A = X$, "on X" is omitted in the statement.

2. Regularized gap functions for (GMWQEP)

In this section, some new gap functions for (GMWQEP) are proposed. Motivated by Mastroeni (2003), the authors consider the following definition of gap functions.

Definition 2.1.

A real valued function $p: A \to \mathbf{R}$ is said to be a *gap function* of (GMWQEP) if it satisfies the following conditions:

(G_1) $p(x) \geq 0,$ for all $x \in K(x);$

(G_2) for any $x_0 \in K(x_0)$, $p(x_0) = 0$ if and only x_0 is a solution of (GMWQEP).

Inspired by the approaches of Yamashita & Fukushima (1997) and Fukushima (1992), the authors develop a regularized gap function for (GMWQEP). Suppose that $K(x)$ is a compact set for each $x \in A$. Then, for each $\theta > 0$, the authors consider a function $\psi_\theta : A \to \mathbf{R}$ defined by

$$\psi_\theta(x) = \max_{y \in K(x)} \{h(x, y) - \theta\pi(x, y)\} \tag{1}$$

where

$$h(x, y) = \min_{1 \leq i \leq m} \{-F_i(x, y) + \langle T_i(x), \eta(x, y)\rangle + \phi_i(x, x) - \phi_i(x, y)\}$$

and $\pi : A \times A \to \mathbf{R}$ is a continuously differentiable function, which has the following property with the associated constants $\delta \geq 2\gamma > 0.$

$$(\Delta_\pi): \text{ For all } x, y \in A,\ \gamma \| x - y \|^2 \leq \pi(x, y) \leq (\delta - \gamma) \| x - y \|^2 . \tag{2}$$

Remark 2.2.

The function ψ_θ in (1) is well-defined. Indeed, as F_i, T_i, ϕ_i and η are continuous for any $i = 1, 2, ..., m$, the function h is continuous. Combine the continuity of h, π and the compactness of $K(x)$ for each $x \in A,$ we have ψ_θ is well-defined.

It will be shown that ψ_θ is a gap function for (GMWQEP) under suitable conditions.

Theorem 2.3.

Assume that

(i) *K has compact and convex values on A;*

(ii) *F_i, ϕ_i and η are convex in the second components for all $i = 1, 2, ..., m$;*

(iii) *π satisfies condition* (Δ_π).

Then, for $\theta > 0$, the function ψ_θ defined by (1) *is a gap function for* (GMWQEP).

Proof.

(G_1) It is clear that for any $x \in K(x)$,

$$\psi_\theta(x) = \max_{y \in K(x)} \{h(x, y) - \theta\pi(x, y)\} \geq h(x, x) - \theta\pi(x, x). \tag{3}$$

We have $\pi(x, x) = 0$ and

$$h(x, x) = \min_{1 \leq i \leq m} \{-F_i(x, x) + \langle T_i(x), \eta(x, x)\rangle + \phi_i(x, x) - \phi_i(x, x)\} = 0.$$

Thus, from (3), it can be concluded that $\psi_\theta(x)\ge 0$ for any $x\in K(x)$.

(G_2) If there exists $x_0\in K(x_0)$ such that $\psi_\theta(x_0)=0$, i.e.,

$$h(x_0,y)-\theta\pi(x_0,y)\le 0,\forall y\in K(x_0) \text{ or}$$

$$\min_{1\le i\le m}\{-F_i(x_0,y)+\langle T_i(x_0),\eta(x_0,y)\rangle+\phi_i(x_0,x_0)-\phi_i(x_0,y)\}\le\theta\pi(x_0,y),\forall y\in K(x_0).$$

For arbitrary $x\in K(x_0)$ and $\lambda\in(0,1)$, let $y_\lambda=\lambda x_0+(1-\lambda)x$. Sine $K(x_0)$ is convex, we get $y_\lambda\in K(x_0)$ and

$$\min_{1\le i\le m}\{-F_i(x_0,y_\lambda)+\langle T_i(x_0),\eta(x_0,y_\lambda)\rangle+\phi_i(x_0,x_0)-\phi_i(x_0,y_\lambda)\}\le\theta\pi(x_0,y_\lambda). \tag{4}$$

Since F_i, ϕ_i and η are convex in the second components for all $i=1,2,...,m$, we have

$$-F_i(x_0,y_\lambda)\ge-\lambda F_i(x_0,x_0)-(1-\lambda)F_i(x_0,x)=-(1-\lambda)F_i(x_0,x), \tag{5}$$

$$\langle T_i(x_0),\eta(x_0,y_\lambda)\rangle\ge\langle T_i(x_0),(1-\lambda)\eta(x_0,x)\rangle=(1-\lambda)\langle T_i(x_0),\eta(x_0,x)\rangle, \tag{6}$$

$$\phi_i(x_0,x_0)-\phi_i(x_0,y_\lambda)\ge(1-\lambda)\phi_i(x_0,x_0)-(1-\lambda)\phi_i(x_0,x). \tag{7}$$

As π satisfies condition (Δ_π), we have

$$\theta(x_0,x+\lambda(x_0-x))\le(\delta-\gamma)\,\|x_0-x-\lambda(x_0-x)\|^2=(1-\lambda)^2(\delta-\gamma)\,\|x_0-x\|^2. \tag{8}$$

From (4)-(8), we get that

$$\min_{1\le i\le m}\{-(1-\lambda)F_i(x_0,x)+(1-\lambda)\langle T_i(x_0),\eta(x_0,x)\rangle+(1-\lambda)\phi_i(x_0,x_0)-(1-\lambda)\phi_i(x_0,x)\}$$
$$\le(1-\lambda)^2(\delta-\gamma)\,\|x_0-x\|^2.$$

Equivalently,

$$(1-\lambda)\min_{1\le i\le m}\{-F_i(x_0,x)+\langle T_i(x_0),\eta(x_0,x)\rangle+\phi_i(x_0,x_0)-\phi_i(x_0,x)\}$$
$$\le(1-\lambda)^2(\delta-\gamma)\,\|x_0-x\|^2.$$

So,

$$\min_{1\le i\le m}\{-F_i(x_0,x)+\langle T_i(x_0),\eta(x_0,x)\rangle+\phi_i(x_0,x_0)-\phi_i(x_0,x)\}\le(1-\lambda)(\delta-\gamma)\,\|x_0-x\|^2. \tag{9}$$

Taking the limit as $\lambda\to 1$ in (9), we obtain

$$\min_{1\le i\le m}\{-F_i(x_0,x)+\langle T_i(x_0),\eta(x_0,x)\rangle+\phi_i(x_0,x_0)-\phi_i(x_0,x)\}\le 0.$$

Then, for any $x\in K(x)$, there exits $1\le i_0\le m$ such that

$$F_{i_0}(x_0,x)+\langle T_{i_0}(x_0),\eta(x,x_0)\rangle+\phi_{i_0}(x_0,x)-\phi_{i_0}(x_0,x_0)\ge 0,$$

that is,

$$F(x_0,x)+\langle T(x_0),\eta(x,x_0)\rangle+\phi(x_0,x)-\phi(x_0,x_0)\notin-\operatorname{int}\mathbf{R}_+^m,\forall x\in K(x).$$

Hence, $x_0\in S$.

Conversely, if $x_0\in S$, then there exists $1\le i_0\le m$ such that

$$F_{i_0}(x_0,y)+\langle T_{i_0}(x_0),\eta(y,x_0)\rangle+\phi_{i_0}(x_0,y)-\phi_{i_0}(x_0,x_0)\geq 0,\forall y\in K(x).$$

This means that

$$\min_{1\leq i\leq m}\{-F_i(x_0,y)+\langle T_i(x_0),\eta_i(x_0,y)\rangle+\phi_i(x_0,x_0)-\phi_i(x_0,y)-\theta\pi(x,y)\}\leq 0,\forall y\in K(x)$$

or

$$\max_{y\in K(x)}\min_{1\leq i\leq m}\{-F_i(x_0,y)+\langle T_i(x_0),\eta_i(x_0,y)\rangle+\phi_i(x_0,x_0)-\phi_i(x_0,y)-\theta\pi(x,y)\}\leq 0.$$

So, $\psi_\theta(x_0)\leq 0$. Since $\psi_\theta(x)\geq 0$ for any $x\in K(x)$, $\psi_\theta(x_0)=0$. This completes the proof. □

Lemma 2.4.

Assume that K is continuous with compact values, F_i,T_i,ϕ_i are continuous for all $i=1,2,...,m$. Then, for each $\theta>0$, ψ_θ is continuous on A.

Proof.

Since F_i,ϕ_i,T_i,η are continuous for all $i=1,...,m$, we get that

$$h(x,y)=\min_{1\leq i\leq m}\{-F_i(x,y)+\langle T_i(x),\eta(x,y)\rangle+\phi_i(x,x)-\phi_i(x,y)\}$$

is continuous for $x,y\in A$. Hence, for each $\theta>0$, $h(x,y)-\theta\pi(x,y)$ is continuous for $x,y\in A$ (since π is continuous). Moreover, K is continuous with compact values on A, so it follows from the Maximum Theorem (Proposition 23 in Aubin & Ekeland (1984),) that ψ_θ defined by

$$\psi_\theta(x)=\min_{x\in K(x)}\{h(x,y)-\theta\pi(x,y)\}$$

is continuous on A. □

Motivated by Yamashita & Fukushima (1997), we propose a gap function base on the Moreau-Yosida regularization of ψ_θ as follows:

$$H_{\psi_\theta,\tau}(x)=\min_{z\in K(x)}\{\psi_\theta(z)+\tau\rho(x,z)\} \tag{10}$$

where $x\in K(x)$, $\tau>0$ and $\rho:A\times A\to\mathbf{R}$ is a continuously different function, which has the following property with the associated constants $b\geq 2a>0$.

(Δ_ρ): for all $x,y\in A$, $a\ \|x-y\|^2\leq\rho(x,y)\leq(b-a)\ \|x-y\|^2$.

We can rewrite $H_{\psi_\theta,\tau}(x)$ as follow:

$$H_{\psi_\theta,\tau}(x)=\min_{z\in K(x)}\left[\max_{y\in K(z)}\{h(z,y)-\theta\pi(z,y)\}+\tau\rho(x,z)\right] \tag{11}$$

Now, it will be proven that $H_{\psi_\theta,\tau}$ ia a gap function for (GMWQEP).

Theorem 2.5.

Assume that all the condition of Theorem 2.3 and Lemma 2.4 hold and assume further that

(i) *for any $x,z\in A$, if $x\in K(x)$ and $z\in K(x)$ then $z\in K(z)$;*

(ii) ρ *satisfies condition* (Δ_ρ).

Then, $H_{\psi_\theta,\tau}$ *defined by* (11) *is gap function for* (GEMVEP).

Proof.

(G_1) For any $\theta,\tau>0$ and $x\in K(x)$. Let $z\in K(x)$ be arbitrary, it follows from the assumption (i) that $z\in K(z)$. Since ψ_θ is a gap function, we have $\psi_\theta(z)\geq 0$. Consequently, $H_{\psi_\theta,\tau}(x)\geq 0$ for all $x\in K(x)$.

(G_2) Suppose that $x_0\in S$. Theorem 2.3 implies that $\psi_\theta(x_0)=0$. Therefore,

$$H_{\psi_\theta,\tau}(x_0)=\min_{z\in K(x_0)}\{\psi_\theta(z)+\tau\rho(x_0,z)\}\leq \psi_\theta(x_0)+\tau\rho(x_0,x_0)=0.$$

Since $H_{\psi_\theta,\tau}(x_0)\geq 0$, we get $H_{\psi_\theta,\tau}(x_0)=0$.

Conversely, if $H_{\psi_\theta,\tau}(x_0)=0$, i.e, $\min_{z\in K(x_0)}\{\psi_\theta(z)+\tau\rho(x_0,z)\}=0$. Then, for each n, there is $z_n\in K(x_0)$ such that

$$\psi_\theta(z_n)+\tau\rho(x_0,z_n)<\frac{1}{n}\ . \tag{12}$$

Since ρ satisfied condition (Δ_ρ), it follows from (12) that

$$0\leq \psi_\theta(z_n)+\tau a\ \|x_0-z_n\ \|^2<\frac{1}{n}$$

and, hence $\psi_\theta(z_n)\to 0$ and $\|x_0-z_n\|\to 0$. Using Lemma 2.4, the continuity of ψ_θ is established and then $\psi_\theta(x_0)=0$. Applying Theorem 2.3, we have $x_0\in S$. This completes the proof. □

Example 2.6.

Let $n=1, m=2, A=[0,1]$, $\theta=1$, $\tau=1/2$, $K(x)=[0,x]$, $T_1(x)=x$, $T_2(x)=2x$, $F_1(x,y)=y^2+3xy-4x^2$, $F_2(x)=y^2+8xy-9x^2$, $\eta(y,x)=y-x$, $\phi_1(x,y)=\phi_2(x,y)=0$, and $\pi(x,y)=\rho(x,y)\}=\|x-y\ \|^2$ for all $x,y\in A$. Then, the problem (GMWQEP) is equivalent to finding $x\in[0,x]\cap[0,1]$ such that

$$\begin{aligned}&F(x,y)+\langle T(x),\eta(y,x)\rangle+\phi(x,y)-\phi(x,x)\\&=((y^2+3xy-4x^2),(y^2+8xy-9x^2))+\langle(x,2x),(y-x)\rangle\\&=((y-x)(5x+y),(y-x)(11x+y))\notin -\operatorname{int}\mathbf{R}^2_+,\forall y\in[0,x].\end{aligned}$$

It follows from some direct computations that $S=\{0\}$.

It is not hard to see that all assumptions imposed in Theorems 2.3 and 2.5 are satisfied. Hence, the functions ψ_θ and $H_{\psi_\theta,\tau}$ defined by (3.1) and (3.11) are gap functions for (GMWQEP), respectively. Indeed,

$$\begin{aligned}\psi_\theta(x) &= \max_{y\in K(x)}\left\{h(x,y)-\theta\pi(x,y)\right\}\\ &= \max_{y\in[0,x]}\left\{\min\{(x-y)(5x+y),(x-y)(11x+y)\}-(x-y)^2\right\}\\ &= \max_{y\in[0,x]}\left\{4x^2-2xy-2y^2\right\}=4x^2,\\ H_{\psi_\theta,\tau}(x) &= \min_{z\in K(x)}\{\psi_\theta(z)+\tau\rho(x,z)\}\\ &= \min_{z\in[0,x]}\{4z^2+\frac{1}{2}(x-z)^2\}\\ &= \min_{z\in[0,x]}\left\{\frac{9}{2}z^2-xz+\frac{1}{2}x^2\right\}=\frac{4}{9}x^2.\end{aligned}$$

Remark 2.7.

(i) In special cases of **(a)-(d)** mentioned in Sect. 1, the function ψ_θ reduces to the regularized gap function for (GEMVEP), (GMVEP), (VIP) and (QEP) considered in Husain & Singh (2017), Khan & Chen (2015), Yamashita & Fukushima (1997), Bigi & Passacantando (2016), respectively. Therefore, for these cases, Theorem 2.3 extends to the existing ones in the literature such as Theorem 3.2 in Husain & Singh (2017), Theorem 3.1 in Khan & Chen (2015), Lemma 2.1 in Yamashita & Fukushima (1997) and Theorem 1 in Bigi & Passacantando (2016).

(ii) To the best of our knowledge, up to now, since the regularized gap functions of Moreau-Yosida type for (GMWQEP) in finite dimensional spaces have not been considered in any work, our result, Theorem 2.5 is an improvement. Moreover, in special case of **(c)** mentioned in Sect. 1, the function $H_{\psi_\theta,\tau}$ reduces to the regularized gap function of Moreau-Yosida type for (VIP) considered in Yamashita & Fukushima (1997). Thus, Theorem 2.5 extends Theorem 2.4 in Yamashita & Fukushima (1997).

3. Error bounds for (GMWQEP)

In this section, error bounds for (GMWQEP) are investigated by using the terms of regularized gap functions in Section 2.

Theorem 3.1.

Let x_0 *be a solution of* (GMWQEP). *Suppose that all the conditions of Theorem 2.3 hold and for each* $i=1,2,...,m$, *let* ϕ_i *be skew-symmetric,* F_i *be strongly monotone with modulus* $\alpha_i>0$ *and* T_i *be* η*-strongly monotone with modulus* $\mu_i>0$. *Let* $\alpha=\min_{1\le i\le m}\alpha_i$ *and* $\mu=\min_{1\le i\le m}\mu_i$. *Assume further that* $\bigcap_{i=1}^m S^i\neq\emptyset$, $x_0\in K(x)$ *for any* $x\in K(x_0)$ *and* $\theta>0$ *satisfying* $\alpha+\mu>\theta(\delta-\gamma)$. *Then, for each* $x\in K(x_0)$,

$$\|x-x_0\|\le\sqrt{\frac{\psi_\theta(x)}{\alpha+\mu-\theta(\delta-\gamma)}}\ . \tag{13}$$

Proof.

Since $\bigcap_{i=1}^{m} S^i \neq \emptyset$, all $(\text{GMWQEP})^i$ have the same solution. Without loss of generality, we assume that x_0 is the same solution. For each $x \in K(x_0)$, we have $x_0 \in K(x)$. This implies

$$\begin{aligned}\psi_\theta(x) &= \max_{y\in K(x)}\left\{\min_{1\le i\le m}\{-F_i(x,y)+\langle T_i(x),\eta(x,y)\rangle+\phi_i(x,x)-\phi_i(x,y)\}-\theta\pi(x,y)\right\}\\ &\ge \min_{1\le i\le m}\{-F_i(x,x_0)+\langle T_i(x),\eta(x,x_0)\rangle+\phi_i(x,x)-\phi_i(x,x_0)\}-\theta\pi(x,x_0).\end{aligned} \quad (14)$$

Without loss of generality, we assume that there exists $i_0 \in [1,m]$ such that

$$\begin{aligned}&\min_{1\le i\le m}\{-F_i(x,x_0)+\langle T_i(x),\eta(x,x_0)\rangle+\phi_i(x,x)-\phi_i(x,x_0)\}\\ &\quad = F_{i_0}(x_0,x)+\langle T_{i_0}(x_0),\eta(x,x_0)\rangle+\phi_{i_0}(x_0,x)-\phi_{i_0}(x_0,x_0).\end{aligned} \quad (15)$$

From (14) and (15), we get

$$\psi_\theta(x) \ge F_{i_0}(x_0,x)+\langle T_{i_0}(x_0),\eta(x,x_0)\rangle+\phi_{i_0}(x_0,x)-\phi_{i_0}(x_0,x_0)-\phi_i(x,x_0)\}-\theta\pi(x,x_0). \quad (16)$$

Since F_{i_0} is strongly monotone with modulus α_{i_0}, we conclude that

$$-F_{i_0}(x_0,x)-F_{i_0}(x,x_0)-\alpha_{i_0}\|x-x_0\|^2 \ge 0. \quad (17)$$

It follows from the η-strong monotonicity of T_{i_0} with modulus μ_{i_0} that

$$\langle T_{i_0}(x),\eta(x,x_0)\rangle-\langle T_{i_0}(x_0),\eta(x,x_0)\rangle-\mu_{i_0}\|x-x_0\|^2 \ge 0. \quad (18)$$

As ϕ_{i_0} is skew-symmetric, we get that

$$\phi_{i_0}(x,x)-\phi_{i_0}(x,x_0)-\phi_{i_0}(x_0,x)+\phi_{i_0}(x_0,x_0) \ge 0. \quad (19)$$

Since $x_0 \in S^{i_0}$,

$$F_{i_0}(x_0,x)+\langle T_{i_0}(x_0),\eta(x,x_0)\rangle+\phi_{i_0}(x_0,x)-\phi_{i_0}(x_0,x_0) \ge 0. \quad (20)$$

Employing (17)-(20), we obtain

$$-F_{i_0}(x,x_0)+\langle T_{i_0}(x),\eta(x,x_0)\rangle+\phi_{i_0}(x,x)-\phi_{i_0}(x,x_0) \ge (\alpha_{i_0}+\mu_{i_0})\|x-x_0\|^2. \quad (21)$$

Moreover, it follows from the property (Δ_π) that

$$-\pi(x,x_0) \ge -(\delta-\gamma)\|x-x_0\|^2. \quad (22)$$

From (16), (21) and (22), we get

$$\psi_\theta(x) \ge (\alpha+\mu-\theta(\delta-\gamma))\|x-x_0\|^2.$$

Therefore,

$$\|x-x_0\| \le \sqrt{\frac{\psi_\theta(x)}{\alpha+\mu-\theta(\delta-\gamma)}}$$

and hence the proof is completed. □

Theorem 3.2.

Let x_0 *be a solution of* (GMWQEP). *Assume that all the conditions of Theorem 2.5 and Theorem 3.1 hold. Then, for any* $x \in K(x_0)$, $\tau > 0$, *we have*

$$\| x - x_0 \| \le \sqrt{\frac{2H_{\psi_\theta,\tau}(x)}{\min\{\alpha+\mu-\theta(\delta-\gamma), \tau a\}}}. \quad (23)$$

Proof.

Thanks to Theorem 3.1, we obtain

$$\begin{aligned} H_{\psi_\theta,\tau}(x) &= \min_{z\in K(x)}\{\psi_\theta(z) + \tau\rho(x,z)\} \\ &\ge \min_{z\in K(x)}\left\{(\alpha+\mu-\theta(\delta-\gamma)) \|x_0 - z\|^2 + \tau a \|z - x\|^2\right\} \\ &\ge \min\{\alpha+\mu-\theta(\delta-\gamma), \tau a\} \min_{z\in K(x)}\left\{\|x_0 - z\|^2 + \|z - x\|^2\right\} \\ &\ge \frac{1}{2}\min\{\alpha+\mu-\theta(\delta-\gamma), \tau a\} \|x - x_0\|^2, \end{aligned}$$

where the following inequality is applied:

$$\|x_0 - z\|^2 + \|z - x\|^2 \ge \frac{(\|x_0 - z\| + \|z - x\|)^2}{2} \ge \frac{\|x - x_0\|^2}{2}.$$

This implies

$$\| x - x_0 \| \le \sqrt{\frac{2H_{\psi_\theta,\tau}(x)}{\min\{\alpha+\mu-\theta(\delta-\gamma), \tau a\}}}.$$

Therefore, the proof is completed. □

Example 3.3.

Let $n, m, A, \theta, \tau, K, T_1, T_2, F_1, F_2, \eta, \phi_1, \phi_2, \pi, \rho$ be as Example 2.6. From Example 2.6, we have $\bigcap_{i=1}^{m} S^i \ne \emptyset = \{0\} = S$ and the gap functions of (GMWQEP) are defined by $\psi_\theta(x) = 4x^2$ and $H_{\psi_\theta,\tau}(x) = \frac{4}{9}x^2$.

It is easy to check that F_1 and F_2 are strongly monotone with moduli $\alpha_1 = 3$ and $\alpha_2 = 8$. Also T_1 and T_2 are η-strongly monotone with the moduli $\mu_1 = 1$ and $\mu_2 = 2$. For that reason, $\alpha = 3, \mu = 1$. Moreover, ϕ_1 and ϕ_2 are skew-symmetric and we also obtain $\delta = b = 2, \gamma = a = 1$. Therefore, the assumptions of Theorems 3.1 and 3.2 are satisfied, and so Theorems 3.1 and 3.2 hold.

Indeed, for all $x \in K(x) = [0, x]$, we have $\|x - x_0\| = x$ and

$$\sqrt{\frac{\psi_\theta(x)}{\alpha+\mu-\theta(\delta-\gamma)}}=\sqrt{\frac{4x^2}{3}}=\sqrt{\frac{4}{3}}x\geq x=\|x-x_0\|,$$

$$\sqrt{\frac{2H_{\psi_\theta,\tau}(x)}{\min\{\alpha+\mu-\theta(\delta-\gamma),\tau a\}}}=\sqrt{\frac{2.\frac{4}{9}x^2}{\min\left\{3,\frac{1}{2}\right\}}}=\frac{4}{3}x\geq x=\|x-x_0\|.$$

Thus, the inequalities (13) and (23) hold.

Remark 3.4.

(i) In special cases of Remark 2.7(i), Theorem 3.1 is a generalization of Theorem 4.1 in Husain & Singh (2017), Theorem 3.2 in Khan & Chen (2015), Lemma 4.1 in Yamashita & Fukushima (1997) and Theorem 8 in Bigi & Passacantando (2016).

(ii) In special cases of Remark 2.7(ii), Theorem 3.2 improve error bounds via the regularized gap functions of Moreau-Yosida type in Husain & Singh (2017), Khan & Chen (2015) and Bigi & Passacantando (2016) and is a generalization of Theorem 4.1 in Yamashita & Fukushima (1997).

❖ ***Conflict of Interest:*** *Authors shave no conflict of interest to declare.*

❖ ***Acknowledgements:*** *This research was supported by Ministry of Education and Training of Vietnam under grant number B2019.SPD02.*

REFERENCES

Husain, S., & Singh, N. (2017). Gap functions and error bounds for generalized extended mixed vector equilibrium problem. *Global Sci-Tech*, *9*, 63-70.

Khan, S. A., & Chen, J. W. (2015). Gap functions and error bounds for generalized mixed vector equilibrium problems. *J Optim Theory Appl*, *166*, 767-776.

Yamashita, Y., & Fukushima, M. (1997). Equivalent unconstrained minimization and global error bounds for variational inequality problems. *SIAM J Control Optim*, *35*, 273-284.

Bigi, G., & Passacantando, M. (2016). Gap functions for quasi-equilibria. *J Global Optim*, *66*, 791-810.

Fukushima, M. (1992). Equivalent differentiable optimization problems and descent methods for asymmetric variational inequality problems. *Math Program*, *53*, 99-110.

Anh, L. Q., Hung, N. V., & Tam, V. M. (2018). Regularized gap functions and error bounds for generalized mixed strong vector quasiequilibrium problems. *Comput Appl Math*, *37*, 5935-5950.

Mastroeni, G. (2003). Gap functions for equilibrium problems. *J Global Optim*, *27*, 411-426.

Rockafellar, R. T., & Wets, R. J. B. (1998). *Variational analysis*. Berlin: Springer.

Aubin, J. P., & Ekeland, I. (1984). *Applied nonlinear analysis*. New York: Wiley.

CẬN SAI SỐ CHO BÀI TOÁN TỰA CÂN BẰNG VÉCTƠ YẾU HỖN HỢP TỔNG QUÁT THÔNG QUA HÀM GAP CHỈNH HÓA

Võ Minh Tâm, Nguyễn Huỳnh Vũ Duy, Nguyễn Kim Phát

Khoa Sư phạm Toán học – Trường Đại học Đồng Tháp

Tác giả liên hệ: Võ Minh Tâm – Email: vmtam@dthu.edu.vn

Ngày nhận bài: 13-12-2018; ngày nhận bài sửa: 01-3-2019; ngày duyệt đăng: 25-3-2019

TÓM TẮT

Trong bài báo này, chúng tôi giới thiệu những hàm gap chỉnh hóa cho một lớp các bài toán tựa cân bằng véctơ yếu hỗn hợp tổng quát. Sau đó, những cận sai số cho lớp các bài toán này cũng được thiết lập thông qua những hàm gap chỉnh hóa. Đồng thời, một số ví dụ được xây dựng để mô tả cho những kết quả đạt được.

Từ khóa: tựa cân bằng véctơ yếu hỗn hợp, đơn điệu mạnh, hàm gap chỉnh hóa, cận sai số.