TẠP CHÍ KHOA HỌC
TRƯỜNG ĐẠI HỌC SƯ PHẠM TP HỒ CHÍ MINH
Tập 21, Số 9 (2024): 1611-1622
Website: https://journal.hcmue.edu.vn

HO CHI MINH CITY UNIVERSITY OF EDUCATION
JOURNAL OF SCIENCE
Vol. 21, No. 9 (2024): 1611-1622
https://doi.org/10.54607/hcmue.js.21.9.4410(2024)

ISSN: 2734-9918

Research Article

BLOW UP IN A NONLINEAR VISCOELASTIC WAVE EQUATION WITH STRONG DAMPING AND VARIABLE SOURCES

Nguyen Van Y

Faculty of Applied Sciences, Ho Chi Minh City University of Industry and Trade, Vietnam

Corresponding author: Nguyen Van Y – Email: ynv@huit.edu.vn

Received: July 07, 2024; Revised: September 16, 2024; Accepted: September 18, 2024

ABSTRACT

This paper is devoted to studying the finite-time blow-up in high initial energy for the solution of the nonlinear viscoelastic wave equation.

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t = |u|^{p(x)-2} u,$$

in a bounded domain $\Omega \subset \mathbb{R}^n$. *Our result improves the blow-up result in the previous work that was obtained by Le et al. (2023).*

Keywords: Blow-up; Nonlinear wave equation; Strong damping; Variable exponent sources; Viscoelasticity

1. Introduction

The aim of this paper is to improve the blow-up result studied by Le et al. (2023) concerning the nonlinear viscoelastic wave equation with firm damping and variable exponents

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + h(u_t) = f(u), & \text{in} \quad \Omega \times (0,T), \\ u = 0 & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where $T > 0$ and $\Omega \subset \mathbb{R}^n (n \geq 2)$ is a bounded domain with smooth boundary $\partial\Omega$, $f(u) = |u|^{p(x)-2} u$, $h(u_t) = -\Delta u_t$, u_0 and u_1 are given initial data, g is a C^1 positive nonincreasing function, the exponents $p(x)$ is continuous on Ω with the logarithmic module of continuity:

$$2 < p^- = \inf_{x \in \Omega} p(x) \leq p(x) \leq p^+ = \sup_{x \in \bar{\Omega}} p(x) \leq \frac{2(n-1)}{n-2}, \quad n \geq 3, \tag{1.2}$$

Cite this article as: Nguyen Van Y (2024). Blow up in a nonlinear viscoelastic wave equation with strong damping and variable sources. *Ho Chi Minh City University of Education Journal of Science*, *21*(9), 1611-1622.

$$\forall x, y \in \Omega, |x-y|<1, |p(x)-p(y)| \le \omega(|x-y|), \tag{1.3}$$

where $\limsup_{\tau \to 0^+} \omega(\tau)\ln(1/\tau) = C < +\infty.$

The problems related to (1.1) arise in many contemporary physical and engineering models, such as electrorheological fluids (smart fluids), fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes through a porous media and image processing. Please refer to Antontsev et al. (2020), Berrimi et al. (2006), Chen et al. (2006), Diening et al. (2011), Messaoudi (2003, 2006), Le et al. (2023), Park et al. (2019), Song et al. (2010, 2014) for more information and applications on the topic.

In case $f(u) = |u|^{p-2} u,$ the problem (1.1) has been widely studied in recent decades, and many authors have discussed results on the existence, nonexistence, and decay of solutions (Messaoudi, 2003, 2006; Song et al., 2010, 2014). When $h(u_t) = |u_t|^{m-2} u_t,$ Messaoudi (2003) considered (1.1) and obtained the finite-time blow-up of solution with negative initial energy. After that, Messaoudi (2006) continued to examine that problem and got the finite-time blow-up of solutions with positive initial energy. When $h(u_t) = -\Delta u_t,$ Song et al. (2010) studied the problem (1.1) and obtained the finite-time blow-up of solutions with positive initial energy. After that, Song et al. (2014) also studied the above problem with high arbitrary initial energy and obtained the finite-time blow-up of solutions.

Recently, Le et al. (2023) considered (1.1) with $f(u) = |u|^{p(x)-2} u,\ \ h(u_t) = -\Delta u_t,$ and obtained the blow-up result for local solutions starting in the potential wells. More precisely, under the conditions of the relaxation g as follows:

(H1) the relaxation $g \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and satisfies

$$g(0) > 0,\ 1 - \int_0^\infty g(s)ds = \ell > 0,\ g'(t) \le 0, \quad \text{for all } t \ge 0,$$

(H2) $\displaystyle\int_0^\infty g(s)ds < \frac{p^-(p^- - 2)}{(p^- - 1)^2}.$

the authors obtained the following theorem.

Theorem 1.1. *[Le et al. (2023), Theorem 2.7] Assume that (1.2)-(1.3) and g satisfies (H1), (H2). Assume further that $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ with $u_0 \in \mathcal{U}_\kappa$ and $E(0) < E_{d_\kappa}$, where $E_{d_\kappa} = \dfrac{\frac{1}{2} - \frac{1}{p^-}}{\frac{1}{2} - \frac{1}{p^+}} d_\kappa$. Then solution $u(t)$ to (1.1) blows up in finite time.*

Here $\mathcal{U}_\kappa = \left\{ u \in H_0^1(\Omega) : J_\kappa(u) < E_{d_\kappa}, I_\kappa(u) < 0 \right\}$, J_κ, I_κ, and d_κ as in section 2.1.

Our aim in this paper is to improve the blow-up result of Le et al. (2023) to the blow-up result with high initial energy.

This paper is organised as follows. In the next section, we will give some preliminaries. In Section 3, we state and prove our main results.

2. Preliminaries

2.1. Modified potential wells

Throughout this paper, we define the functionals J_δ, I_δ (for $0<\delta\le\ell$) as in Le et al. (2023) and Nguyen et al. (2023)

$$J_\delta(u)=\frac{\delta}{2}\|\nabla u\|^2-\int_\Omega\frac{|u|^{p(x)}}{p(x)}dx,\quad\text{and}\quad I_\delta(u)=\delta\ \|\nabla u\|^2-\int_\Omega|u|^{p(x)}dx,$$

the Nehari manifold

$$\mathcal{N}_\delta=\left\{u\in H_0^1(\Omega)\setminus\{0\}:I_\delta(u)=\langle J'_\delta(u),u\rangle=0\right\},$$

the potential well-depth

$$d_\delta=\inf_{u\in\mathcal{N}_\delta}J_\delta(u), \tag{2.1}$$

and the modified unstable set as in Le et al. (2023) and Nguyen et al. (2023)

$$\mathcal{U}_\delta=\left\{u\in H_0^1(\Omega):J_\delta(u)<d_\delta,I_\delta(u)<0\right\}.$$

2.2. Definition and preparing results

Let us now define weak solutions to (1.1).

Definition 2.2. *Let $0<T\le\infty$, a function u be called a weak solution of problem (1.1) on $\Omega\times(0,T)$ if*

$$u\in C([0,T);H_0^1(\Omega)),\ u_t\in C([0,T);L^2(\Omega))\cap L^2([0,T);H_0^1(\Omega)),$$

satisfies $u(x,0)=u_0(x)\in H_0^1(\Omega)$, $u_t(x,0)=u_1(x)\in L^2(\Omega)$ and the equality

$$\langle u_{tt},\varphi\rangle+\langle\nabla u,\nabla\varphi\rangle-\int_0^t g(t-s)\langle\nabla u(s),\nabla\varphi\rangle ds+\langle\nabla u_t,\nabla\varphi\rangle=\left\langle|u|^{p(\cdot)-2}u,\varphi\right\rangle,$$

holds for a.e. $t\in(0,T)$ and any $\varphi\in H_0^1(\Omega)$.

Define the energy function

$$E(t)=\frac{1}{2}\|u_t(t)\|^2+\frac{1}{2}\left(1-\int_0^t g(\tau)d\tau\right)\|\nabla u(t)\|^2+\frac{1}{2}(g\circ\nabla u)(t)-\int_\Omega\frac{1}{p(x)}|u(t)|^{p(x)}dx,$$

where $(g\circ\nabla u)(t)=\int_0^t g(t-s)\|\nabla u(t)-\nabla u(s)\|^2ds.$ By testing (1.1) by u_t, we have

$$\frac{d}{dt}E(t)=\frac{1}{2}(g'\circ\nabla u)(t)-\frac{1}{2}g(t)\|\nabla u(t)\|^2-\ \|\nabla u_t(t)\|^2\le 0, \tag{2.2}$$

which implies that $E(t)$ is a non-creasing functional.

We now state the local existence of a solution to (1.1).

Theorem 2.3. *(Local existence) [Le et al. (2023)] Assume that (1.2) – (1.3) and (H1) hold. Then for given $(u_0,u_1)\in H_0^1(\Omega)\times L^2(\Omega)$ the problem (1.1) admits a unique local solution*

$u \in C([0,T_{\max}); H_0^1(\Omega)),\ u_t \in C([0,T_{\max}); L^2(\Omega)) \cap L^2([0,T_{\max}); H_0^1(\Omega)),$

where $T_{\max} > 0$ *is the maximal existence time of* $u(t)$.

We ended this section with the following proposition, which is essential for proving our main results.

Proposition 2.4. *Let (1.2) – (1.3) and (H1) hold and* $0 < \delta \le \ell$. *Let* $u(t) := u(x,t)$ *be a local solution to the problem (1.1). Then, if there exists a time* $t_0 \in [0,T_{\max})$ *such that* $u(t_0) \in \mathcal{U}_\delta$ *and* $E(t_0) < d_\delta$, *then* $u(t)$ *remains inside the set* $\mathcal{U}_\delta$ *for any* $t \in [t_0, T_{\max})$.

Proof. By contradiction, we assume that $u(t)$ leaves $\mathcal{U}_\delta$ at time $t = t_*$, then there exists a sequence $\{t_n\}, t_n \to t_*$ such that $I(u(t_n)) \le 0.$ By the lower semicontinuity, we get $I_\delta(u(t_*)) \le \liminf_{n\to\infty} I_\delta(u(t_n)) \le 0.$

Since $u(t_*) \notin \mathcal{U}_\delta$, we obtain $I_\delta(u(t_*)) = 0$. By variational definition of d_δ, this leads to a contradiction.

Lemma 2.5. *[*Kalantarov et al. (1978)*] Suppose that* $\Phi(t) \in C^2[0,\infty)$ *is a positive function satisfying the following inequality*

$$\Phi(t)\Phi''(t) - (1+\gamma)\left(\Phi'(t)\right)^2 \ge 0,$$

where $\gamma > 0$ *are constants. If* $\Phi(0) > 0,\ \Phi'(0) > 0,$ *then* $\Phi(t) \to \infty$ *for* $t \to t_* \le t^* = \dfrac{\Phi(0)}{\gamma\Phi'(0)}.$

3. Blow-Up

First, we state a theorem about the finite-time blow-up of the solution, whose proof is similar to the proof of Theorem 2.7 in Le et al. (2023) with slightly different, so we omit it here.

Theorem 3.1. *Assume that (1.2)-(1.3) hold and* g *satisfies (H1) and (H2). Assume further that* $(u_0,u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *with* $u_0 \in \mathcal{U}_\kappa$ *and* $E(0) < d_\kappa$, *where*

$$0 < \kappa = \ell - \frac{1}{p^-(p^- - 2)} \int_0^\infty g(s)ds.$$

Then solution $u(t)$ *to (1.1) blows up in finite time; that is, the maximum existence time* T^* *of* u *is finite and*

$$\lim_{t\to T^{*-}} \left(\|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2 ds \right) = +\infty.$$

We can now provide a crucial lemma that supports our main results. The following is the important lemma:

Lemma 3.2. *Let (1.2)-(1.3) and (H1), (H2) hold. Assume that the initial data* $(u_0,u_1) \in H_0^1(\Omega) \times L^2(\Omega)$ *and satisfies*

$$0 < E(0) < \frac{C}{p^-}\int_\Omega u_0 u_1 dx. \tag{3.1}$$

Then the weak solution u *to problem (1.1) satisfies*

$$\int_\Omega u u_t dx - \frac{p^-}{C}E(t) \geq \left(\int_\Omega u_0 u_1 dx - \frac{p^-}{C}E(0)\right)e^{Ct} > 0, \tag{3.2}$$

for all $t \in [0, T),$ *where*

$$C = \min\left\{2 + p^-, \frac{2\lambda_1\left[p^-(p^- - 2) - (p^- - 1)^2(1-\ell)\right]}{2p^- + \lambda_1}\right\}. \tag{3.3}$$

Proof. Using the first equation of (1.1) and integral by parts, we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega u u_t dx &= \int_\Omega u_{tt} u dx + \|u_t(t)\|^2 \\ &= \|u_t(t)\|^2 - \left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|^2 + \int_\Omega |u(t)|^{p(x)}\, dx \\ &\quad - \langle \nabla u_t(t), \nabla u(t)\rangle - \int_0^t g(t-s)\langle \nabla u(t) - \nabla u(s), \nabla u(t)\rangle ds. \end{aligned} \tag{3.4}$$

By using Young inequality, we get

$$-\langle \nabla u_t(t), \nabla u(t)\rangle \geq -\frac{C}{4p^-}\|\nabla u(t)\|^2 - \frac{p^-}{C}\|\nabla u_t(t)\|^2, \tag{3.5}$$

$$\begin{aligned}&-\int_0^t g(t-s)\langle \nabla u(t) - \nabla u(s), \nabla u(t)\rangle ds \\ &\quad \geq -\frac{1}{2p^-}\left(\int_0^t g(s)ds\right)\|\nabla u(t)\|^2 - \frac{p^-}{2}(g \circ \nabla u)(t).\end{aligned} \tag{3.6}$$

Combining (3.5) and (3.6) with (3.4), and using the definition of $E(t),$ and the condition (H1), (H2), we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega u u_t dx &\geq \|u_t(t)\|^2 - \left[1 - \int_0^t g(s)ds - \frac{1}{2p^-}\int_0^t g(s)ds - \frac{C}{4p^-}\right]\|\nabla u(t)\|^2 \\ &\quad + \int_\Omega |u(t)|^{p(x)}\, dx - \frac{p^-}{C}\|\nabla u_t(t)\|^2 - \frac{p^-}{2}(g \circ \nabla u)(t) \\ &\geq \left(1 + \frac{p^-}{2}\right)\|u_t(t)\|^2 + \left[\left(\frac{p^-}{2} - 1\right) - \left(\frac{p^-}{2} - 1 + \frac{1}{2p^-}\right)\int_0^t g(s)ds - \frac{C}{4p^-}\right]\|\nabla u(t)\|^2 \\ &\quad - \frac{p^-}{C}\|\nabla u_t(t)\|^2 - p^- E(t) + \int_\Omega \left(1 - \frac{p^-}{p(x)}\right)|u(t)|^{p(x)}\, dx \\ &\geq \left(1 + \frac{p^-}{2}\right)\|u_t(t)\|^2 + \left[\left(\frac{p^-}{2} - 1\right) - \left(\frac{p^-}{2} - 1 + \frac{1}{2p^-}\right)(1-\ell) - \frac{C}{4p^-}\right]\|\nabla u(t)\|^2 \\ &\quad - p^- E(t) - \frac{p^-}{C}\|\nabla u_t(t)\|^2. \end{aligned} \tag{3.7}$$

Recalling (3.3), then

$$C \le \frac{2\lambda_1\left[p^-(p^- - 2) - (p^- - 1)^2(1-\ell)\right]}{2p^- + \lambda_1} \le 2\left[p^-(p^- - 2) - (p^- - 1)^2(1-\ell)\right].$$

Therefore, it implies from (3.7) that

$$\frac{d}{dt}\int_\Omega u u_t dx \ge \left(1+\frac{p^-}{2}\right)\|u_t(t)\|^2 + \left[\left(\frac{p^-}{2}-1\right)-\left(\frac{p^-}{2}-1+\frac{1}{2p^-}\right)(1-\ell)-\frac{C}{4p^-}\right]\lambda_1\|u(t)\|^2$$
$$- p^-E(t) - \frac{p^-}{C}\|\nabla u_t(t)\|^2, \qquad (3.8)$$

Here we used the inequality $\lambda_1\|u\|^2 \le \|\nabla u\|^2$.

Put

$$H(t) = \int_\Omega u u_t dx - \frac{p^-}{C}E(t).$$

Combining (3.8) with (2.2), we obtain

$$\frac{d}{dt}H(t) \ge \left(1+\frac{p^-}{2}\right)\|u_t(t)\|^2$$
$$+\left[\left(\frac{p^-}{2}-1\right)-\left(\frac{p^-}{2}-1+\frac{1}{2p^-}\right)(1-\ell)-\frac{C}{4p^-}\right]\lambda_1\|u(t)\|^2 - p^-E(t)$$
$$\ge C\left(\frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|u(t)\|^2 - \frac{p^-}{C}E(t)\right) \ge CH(t) \qquad (3.9)$$

according to (3.3). Furthermore, it follows from (3.1) that

$$H(0) = \int_\Omega u_0 u_1 dx - \frac{p^-}{C}E(0) > 0.$$

By Gronwall's inequality, we deduce from (3.9) that

$$H(t) \ge e^{Ct}H(0) > 0.$$

The proof is complete.

We then provide and validate the finite-time blow-up results for solutions with high initial energy and estimate and provide an upper bound on the blow-up time.

Theorem 3.3. *(Finite time blow-up for high initial energy) Let all the assumptions in Lemma 3.2 be fulfilled. Then the solution u for problem (1.1) blows up in finite time.*

Proof. By contradiction, we assume that $u(t)$ exists globally.

First, by using Holder's inequality and (2.2), we get

$$\|u(t)\| = \left\|u_0(\cdot)\right\| + \int_0^t u_\tau(\tau)d\tau \le \|u_0\| + \int_0^t \|u_\tau(\tau)\| d\tau$$
$$\le \|u_0\| + \frac{1}{\sqrt{\lambda_1}}\int_0^t \|\nabla u_\tau(\tau)\| d\tau \le \|u_0\| + \frac{\sqrt{t}}{\sqrt{\lambda_1}}\left(\int_0^t \|\nabla u_\tau(\tau)\|^2 d\tau\right)^{1/2}$$
$$\le \|u_0\| + \frac{\sqrt{t}}{\sqrt{\lambda_1}}\left(E(0) - E(t)\right)^{1/2}, \tag{3.10}$$

for all $t \in [0,\infty)$. Since u is a global solution of problem (1.1), then $E(t) \ge 0,$ for all $t \in [0,\infty)$. Otherwise, there exists $t_0 \in [0,\infty)$ so that $E(t_0) < 0$. By virtue of the definition of $E(t),\ J_\delta,\ I_\delta$ and

$$J_\delta(u) \ge \left(\frac{1}{2} - \frac{1}{p^-}\right)\delta \|\nabla u\|^2 + \frac{1}{p^-} I_\delta(u),$$

we obtain

$$E(t) \ge \frac{1}{2}\|u_t(t)\|^2 + J_\delta(u(t)) + \frac{1}{2}(g \circ \nabla u)(t)$$
$$\ge \frac{1}{2}\|u_t(t)\|^2 + \left(\frac{1}{2} - \frac{1}{p^-}\right)\delta \ \|\nabla u(t)\|^2 + \frac{1}{p^-} I_\delta(u(t)) + \frac{1}{2}(g \circ \nabla u)(t). \tag{3.11}$$

Since $p^- > 2$ and the assumption (H1), we get that $E(t_0) < 0$ implies $I_\delta(u(x,t_0)) \le 0$ due to (3.11). By choosing $u(x,t_0)$ as the new initial data, Theorem 3.1 indicates that u blows up in finite time, which is a contradiction. Thus, due to (2.2), we get $0 \le E(t) \le E(0)$. Therefore, we deduce from (3.10) that for all $t \in [0,\infty)$,

$$\|u(t)\| \le \|u_0\| + \frac{\sqrt{t}}{\sqrt{\lambda_1}}\left(E(0)\right)^{1/2}. \tag{3.12}$$

On the other hand, by using (3.2) one gets

$$\frac{d}{dt}\|u(t)\|^2 = 2\int_\Omega u(t)u_t(t)dx \ge 2H(0)e^{Ct} + \frac{2p}{C}E(t) \ge 2H(0)e^{Ct} > 0. \tag{3.13}$$

Integrating (3.13) from 0 to t, we obtain

$$\|u(t)\|^2 = \|u_0\|^2 + 2\int_0^t \int_\Omega u(\tau)u_\tau(\tau)dxd\tau$$
$$\ge \|u_0\|^2 + 2\int_0^t H(0)e^{C\tau}d\tau = \|u_0\|^2 + \frac{2}{C}\left(e^{Ct} - 1\right)H(0), \tag{3.14}$$

which contradicts (3.12) for t sufficiently large. Thus, the solution u for the problem (1.1) blows up in finite time.

Theorem 3.4. *(Upper bound of the blow-up time) Let all the assumptions in Lemma 3.2 be fulfilled. In addition, if*

$$E(0) \le \frac{C}{2p^-}\|u_0\|^2, \tag{3.15}$$

then the solution u for the problem (1.1) blows up at some finite time T^* in the sense of

$$\lim_{t\to T^{*-}}\left(\|u(t)\|^2+\int_0^t\|\nabla u(s)\|^2\,ds\right)=+\infty.$$

Moreover, the upper bound for blow-up time T^* is given by

$$T^*\le\frac{2\beta T_0^2+2\|u_0\|^2}{(p^- -2)\left(\int_\Omega u_0u_1dx+\beta T_0\right)-2\|\nabla u_0\|^2},\tag{3.16}$$

where C is the positive constant given by (3.3), $\beta=\dfrac{-2p^-E(0)+C\|u_0\|^2}{2p}$, and T_0 is chosen large enough such that

$$(p^- -2)\left(\int_\Omega u_0u_1dx+\beta T_0\right)-2\|\nabla u_0\|^2>0.\tag{3.17}$$

Proof. First, Theorem 3.3 indicates that the solution u for the problem (1.1) blows up in finite time. Suppose that the blow-up time is T^*. We now give an estimate for the upper bound of T^*.

Define the auxiliary functional

$$\theta(t)=\|u(t)\|^2+\int_0^t\|\nabla u(s)\|^2\,ds+(T^*-t)\|\nabla u_0\|^2+\beta\left(t+T_0\right)^2,\ t\in[0,T^*).\tag{3.18}$$

Then we have

$$\begin{aligned}\theta'(t)&=2\int_\Omega u(t)u_t(t)dx+\|\nabla u(t)\|^2-\|\nabla u_0\|^2+2\beta(t+T_0)\\&=2\langle u_t(t),u(t)\rangle+\int_0^t\frac{d}{ds}\|\nabla u(s)\|^2\,ds+2\beta(t+T_0)\\&=2\langle u_t(t),u(t)\rangle+2\int_0^t\langle\nabla u_t(s),\nabla u(s)\rangle ds+2\beta(t+T_0).\end{aligned}\tag{3.19}$$

It follows from (3.19) that

$$\frac{1}{4}\left(\theta'(t)\right)^2=\Theta(t)+\left(\|u_t(t)\|^2+\int_0^t\|\nabla u_t(s)\|^2\,ds+\beta\right)\left(\|u(t)\|^2+\int_0^t\|\nabla u(s)\|^2\,ds+\beta(t+T_0)^2\right),$$

where $\Theta(t)$ is given by

$$\begin{aligned}\Theta(t)=&\left(\langle u_t(t),u(t)\rangle+\int_0^t\langle\nabla u_t(s),\nabla u(s)\rangle ds+\beta(t+T_0)\right)^2\\&-\left(\|u_t(t)\|^2+\int_0^t\|\nabla u_t(s)\|^2\,ds+\beta\right)\left(\|u(t)\|^2+\int_0^t\|\nabla u(s)\|^2\,ds+\beta(t+T_0)^2\right).\end{aligned}$$

By virtue of the Cauchy-Schwarz inequality, it is not difficult to see that $\Theta(t)\le 0$ for $t\in[0,T^*)$. Hence, we have

$$\begin{aligned}\frac{1}{4}\left(\theta'(t)\right)^2&\le\left(\|u_t(t)\|^2+\int_0^t\|\nabla u_t(s)\|^2\,ds+\beta\right)\left(\|u(t)\|^2+\int_0^t\|\nabla u(s)\|^2\,ds+\beta(t+T_0)^2\right)\\&\le\theta(t)\left(\|u_t(t)\|^2+\int_0^t\|\nabla u_t(s)\|^2\,ds+\beta\right).\end{aligned}\tag{3.20}$$

On the other hand, it implies from (3.19) that

$$\theta''(t) = 2\|u_t(t)\|^2 + 2\langle u_{tt}(t), u(t)\rangle + 2\langle \nabla u_t(t), \nabla u(t)\rangle + 2\beta,$$

which in turn, by multiplying (1.1) by $u(t)$ and integrating over Ω, yields

$$\begin{aligned}\theta''(t) = 2\|u_t(t)\|^2 - 2\left(a - \int_0^t g(s)ds\right)\|\nabla u(t)\|^2 + 2\int_\Omega |u(t)|^{p(x)}\,dx \\ - 2\int_0^t g(t-s)\langle \nabla u(t) - \nabla u(s), \nabla u(t)\rangle ds + 2\beta.\end{aligned} \quad (3.21)$$

It follows from (3.18), (3.20) and (3.21) that

$$\theta''(t)\theta(t) - \frac{p^- + 2}{4}\left(\theta'(t)\right)^2 \geq \theta(t)\zeta(t), \quad (3.22)$$

where $\zeta : [0, T^*) \to \mathbb{R}$ is the function defined by

$$\begin{aligned}\zeta(t) = -p^-\|u_t(t)\|^2 - (p^- + 2)\int_0^t \|\nabla u_t(s)\|^2 ds - 2\left(a - \int_0^t g(s)ds\right)\|\nabla u(t)\|^2 \\ + 2\int_\Omega |u(t)|^{p(x)}\,dx - 2\int_0^t g(t-s)\langle \nabla u(t) - \nabla u(s), \nabla u(t)\rangle ds - p^-\beta.\end{aligned}$$

On the other hand, by using the Cauchy-Schwartz inequality, we have

$$2\int_0^t g(t-s)\langle \nabla u(t) - \nabla u(s), \nabla u(t)\rangle ds \leq \eta\left(g \circ \nabla u\right)(t) + \frac{1}{\eta}\left(\int_0^t g(s)ds\right)\|\nabla u(t)\|^2$$

which implies

$$\begin{aligned}\zeta(t) \geq -p^-\|u_t(t)\|^2 - \eta\left(g \circ \nabla u\right)(t) - (p^- + 2)\int_0^t \|\nabla u_t(s)\|^2 ds \\ - 2\left[1 - \left(1 - \frac{1}{2\eta}\right)\int_0^t g(s)ds\right]\|\nabla u(t)\|^2 + 2\int_\Omega |u(t)|^{p(x)}\,dx - p^-\beta,\end{aligned} \quad (3.23)$$

for any $\eta > 0$. By the definition of $E(t)$ we have

$$\|u_t(t)\|^2 = 2E(t) - \left(1 - \int_0^t g(s)ds\right)\|\nabla u(t)\|^2 - (g \circ \nabla u)(t) + 2\int_\Omega \frac{1}{p(x)}|u(t)^{p(x)}dx. \quad (3.24)$$

Combining (3.23)-(3.24) with the energy inequality (2.2), one has

$$\begin{aligned}\zeta(t) \geq -2p^-E(0) + (p^- - 2)\int_0^t \|\nabla u_t(s)\|^2 ds \\ + \left[(p^- - 2) - \left(p^- - 2 + \frac{1}{\eta}\right)\int_0^t g(s)ds\right]\|\nabla u(t)\|^2 \\ + (p^- - \eta)(g \circ \nabla u)(t) + 2\int_\Omega \left(1 - \frac{p^-}{p(x)}\right)|u(t)|^{p(x)}\,dx - p^-\beta.\end{aligned} \quad (3.25)$$

Since $p^- > 2$, by choosing $\eta = p^-$, it follows from (3.25) that

$$\zeta(t) \geq -2p^- E(0) + \left[(p^- - 2) - \left(p^- - 2 + \frac{1}{p^-} \right) \int_0^t g(s)ds \right] \|\nabla u(t)\|^2 - p^- \beta$$
$$\geq -2p^- E(0) + \left[(p^- - 2) - \left(p^- - 2 + \frac{1}{p^-} \right)(1-\ell) \right] \|\nabla u(t)\|^2 - p^- \beta$$
$$\geq -2p^- E(0) + \left[(p^- - 2) - \left(p^- - 2 + \frac{1}{p^-} \right)(1-\ell) \right] \lambda_1 \|u(t)\|^2 - p^- \beta. \qquad (3.26)$$

Notice that (3.13) implies

$$\|u(t)\|^2 \geq \|u_0\|^2, \text{ for } t \in [0, T^*). \qquad (3.27)$$

From (3.15), (3.22), (3.26) and (3.27), we obtain

$$\theta''(t)\theta(t) - \frac{p^- + 2}{4}(\theta(t))^2$$
$$\geq \theta(t) \left\{ -2p^- E(0) + \left[(p^- - 2)a - \left(p^- - 2 + \frac{1}{p^-} \right)(a-\ell) \right] \lambda_1 \|u_0\|^2 - p^- \beta \right\}$$
$$\geq \theta(t) \left[-2p^- E(0) + C\|u_0\|^2 - p^- \beta \right] \geq 0, \text{ for } t \in [0, T^*), \qquad (3.28)$$

here we used $\beta = \frac{-2p^- E(0) + C\|u_0\|^2}{2p}$ and (3.15) in the last step.

By the definition of $\theta(t)$, it is obviously that $\theta(0) > 0$ for any $T_0 > 0$. In addition, if T_0 is chosen large enough in (3.17) then $\theta'(0) = 2\int_\Omega u_0 u_1 dx + 2\beta T_0 > 0$. Applying Lemma 2.5 with $\gamma = (p^- - 2)/4$ we have that $\theta(t) \to \infty$ for $t \to T^*$, where T^* holds

$$T^* \leq \frac{4\theta(0)}{(p^- - 2)\theta'(0)} = \frac{2\|u_0\|^2 + 2T^* \|\nabla u_0\|^2 + 2\beta T_0^2}{(p^- - 2)\beta T_0 + (p^- - 2)\int_\Omega u_0 u_1 dx},$$

which in turn implies that

$$T^* \leq \frac{2\beta T_0^2 + 2\|u_0\|^2}{(p^- - 2)\left(\int_\Omega u_1 u_0 dx + \beta T_0\right) - 2\|\nabla u_0\|^2}. \qquad (3.29)$$

- ***Conflict of Interest:*** *Author have no conflict of interest to declare.*
- ***Acknowledgement:*** *This work is supported by Ho Chi Minh City University of Industry and Trade (HUIT)*

REFERENCES

Antontsev, S., & Shmarev, S. (2020). *Evolution PDEs with Nonstandard Growth Conditions: Existence, Uniqueness, Localization, Blow-up* (Vol 4). Atlantis Studies in Differential Equations, Atlantis Press, 2015. https://doi.org/10.2991/978-94-6239-112-3

Berrimi, S., & Messaoudi, S. A. (2006). Existence and decay of solutions of a viscoelastic equation with a nonlinear source. *Nonlinear Anal., 64*, 2314-2331. https://doi.org/10.1016/j.na.2005.08.015

Chen, Y., Levine, S., & Rao, M. (2006). Variable exponent, linear growth functionals in image restoration. *SIAM J. Appl. Math., 66*(4), 1383-1406. https://doi.org/10.1137/050624522

Diening L., Harjulehto, P., Hasto, P., & Ruzicka, M. (2011). *Lebesgue and Sobolev spaces with variable exponents*. Lecture Notes in Mathematics 2017, Springer-Verlag, Heidelberg. https://doi.org/10.1007/978-3-642-18363-8

Kalantarov, V. K., & Ladyzhenskaya, O. A. (1978). The occurrence of collapse for quasilinear equations of parabolic and hyperbolic type. *J. Soviet Math*., *10*, 53-70. https://doi.org/10.1007/BF01109723

Le, C. N., Le, X. T., & Nguyen, V. Y. (2023). Exponential decay and blow-up results for a viscoelastic equation with variable sources. *Appl. Anal*., *102*(3), 782-799. https://doi.org/10.1080/00036811.2021.1965581

Messaoudi, S. A. (2003). Blow up and global existence in nonlinear viscoelastic wave equations. *Math. Nachrich, 260*, 58-66. https://doi.org/10.1002/mana.200310104

Messaoudi, S. A. (2006). Blow up of solutions with positive initial energy in a nonlinear viscoelastic wave equation. *J. Math. Anal. Appl., 320*, 902-915. https://doi.org/10.1016/j.jmaa.2005.07.022

Nguyen, V. Y., Le, C. N., & Le, X. T. (2023). On a thermo-viscoelastic system with variable exponent sources. *Nonlinear Anal. RWA*, 71, Article 103807. https://doi.org/10.1016/j.nonrwa.2022.103807

Park, S. H., & Kang, J. R. (2019). *Blow-up of solutions for a viscoelastic wave equation with variable exponents*. *Math. Methods. Appl. Sci., 42*(6), 2083-2097. https://doi.org/10.1002/mma.5501

Song, H., & Zhong, C. (2010). *Blow-up of solutions of a nonlinear viscoelastic wave equation. Nonlinear Anal. RWA*, *11*, 3877-3883. https://doi.org/10.1016/j.nonrwa.2010.02.015

Song, H., & Xue, D. (2014). *Blow up in a nonlinear viscoelastic wave equation with strong damping. Nonlinear Anal.*, *109*, 245-251. https://doi.org/10.1016/j.na.2014.06.012

BÙNG NỔ CHO MỘT PHƯƠNG TRÌNH SÓNG ĐÀN HỒI NHỚT PHI TUYẾN VỚI NGUỒN CHỨA SỐ MŨ BIẾN

Nguyễn Văn Ý

Khoa Khoa học Ứng dụng, Trường Đại học Công thương Thành phố Hồ Chí Minh, Việt Nam

Tác giả liên hệ: Nguyễn Văn Ý – Email: ynv@huit.edu.vn

Ngày nhận bài: 07-7-2024; ngày nhận bài sửa: 16-9-2024; ngày duyệt đăng: 18-9-2024

TÓM TẮT

Bài báo này tập trung nghiên cứu sự bùng nổ trong thời gian hữu hạn của nghiệm với năng lượng ban đầu cao cho một phương trình sóng đàn hồi nhớt phi tuyến

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds - \Delta u_t = |u|^{p(x)-2}u,$$

trên miền bị chặn $\Omega \subset \mathbb{R}^n$. *Các kết quả của chúng tôi cải thiện kết quả bùng nổ trong công trình trước đây được thực hiện bởi Nhàn và các cộng sự (2023).*

Từ khóa: bùng nổ; phương trình sóng phi tuyến; tắt dần mạnh; nguồn có số mũ thay đổi; tính đàn hồi nhớt