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Research Article

THE BOUNDEDNESS OF CALDERÓN-ZYGMUND COMMUTATORS OF TYPE THETA OVER SPACES OF HOMOGENEOUS TYPE

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ABSTRACT

In this paper, we study the boundedness of Calderón-Zygmund operator T *and its commutator* $[b,T]$ *of type* θ *on generalized Morrey-Lorentz spaces* $M_{\varphi}^{p,r}(X)$*, where* X *is a space of homogeneous type. We first modify a well-known result of Calderón-Zygmund decomposition to prove that Calderón-Zygmund operators are of strong type* (p,p) *on* $L^{p}(X)$*, for every* $p\in(1,\infty)$ *(see Lemma 2.1, Lemma 2.2, and Lemma 2.3). By using Kolmogorov's inequality, conditions of Calderón-Zygmund operators, and BMO spaces, we establish two pointwise estimates for sharp maximal operators (see Lemma 2.6 and Lemma 2.8). Finally, we derive the boundedness of* T *and* $[b,T]$ *on* $M_{\varphi}^{p,r}(X)$ *(see Theorem 3.1).*

Keywords: Calderón-Zygmund commutator of type θ; generalised Morrey-Lorentz space; space of homogeneous type

1. Introduction

Since its first introduction by Coifman et al. (1978), the theory of Calderón-Zygmund operators and their commutators has played an important role in modern harmonic analysis and differential and partial differential equations. Several studies have examined the boundedness of Calderón-Zygmund operators and their commutators. Grafakos (2009) indicated that Calderón-Zygmund operators are bounded on L^{p} for $1<p<\infty$ and bounded from L^{1} to $L^{1,\infty}$. Carro et al. (2021) showed that Calderón-Zygmund operators and their commutators are bounded on weighted Lorentz space. The generalized results were also proved in spaces of homogenous type by Dao et al. (2021). Later, Minh et al. (2022) proved

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the boundedness of Calderón-Zygmund operators and commutators of type theta on weighted Lorentz spaces. Then Nghia et al. (2023) showed that Calderón-Zygmund commutators of type theta are bounded on Morrey-Lorentz spaces over $\mathbb{R}^n$.

Motivated by the results mentioned above, in this paper, we aim to extend the boundedness of Calderón-Zydmund operators of type theta and their commutators on generalized Morrey-Lorentz spaces of homogeneous type concerning a doubling measure μ, that is, there exists a constant $C>0$ depending only on μ such that $\mu(2B)\le C\mu(B)$, for every ball $B\subset X$.

Definition 1.1. *A quasi-metric d on a set X is a function $d: X\times X\to[0,+\infty)$ satisfying*

(i) $d(x,y)=d(y,x)$ for all $x,y\in X$;

(ii) $d(x,y)=0$ if and only if $x=y$;

(iii) There exists a constant $A_0\in[1,+\infty)$ such that for every $x,y,z\in X$, the following quasi-triangle inequality holds true

$$d(x,y)\le A_0\left(d(x,z)+d(z,y)\right).$$

We call (X,d,μ) a space of homogeneous type, where d is a quasi-metric and μ is a doubling measure on X.

Note that the constant A_0 in the quasi-triangle inequality above is used throughout the paper without further explanation. For convenience, we denoted $A\lesssim B$ if there exists a positive constant C such that $A\le CB$. Throughout this paper, we assume that $\mu(X)=\infty$ and $\mu(\{x_0\})=0$ for every $x_0\in X$.

In the sequel, we present generalised Morrey-Lorentz and BMO spaces definitions.

Definition 1.2. *Let $p\in(1,\infty)$, $r\in[1,\infty)$. Suppose that f is a real-value function f on X and $\varphi(t):(0;\infty)\to(0;\infty)$ satisfies the following conditions:*

$$\begin{cases} i)\ \varphi(t)\ \text{is nonincreasing} \\ ii)\ \mu(B_t)\varphi^p(t)\ \text{is nondecreasing, for any ball } B_t\subset X, \\ iii)\ \varphi(2t)\le D\varphi(t),\ \forall t>0, \end{cases}$$

(1.1)

for some constant $0<D<1$.

We denote

$$\|f\|_{\mathbf{M}^{p,r}_{\varphi}}=\sup_{B(x,t)}\frac{\|f\|_{L^{p,r}(B(x,t))}}{\mu(B(x,t))^{1/p}\varphi(t)}, \tag{1.2}$$

where the supremum is taken over all the balls $B(x,t)$ *in* X *and* $\|f\|_{L^{p,r}(B(x,t))}$ *denotes the Lorentz norm of* f *on* $B(x,t)$. *Then the Morrey-Lorentz space is defined by*

$$M_{\varphi}^{p,r}(X)=\left\{f\in L^1_{loc}(X):\|f\|_{M_{\varphi}^{p,r}}<\infty\right\}.$$

Definition 1.3. *Let* $f\in L^1_{loc}(X)$. *We denote*

$$\|f\|_*=\sup_Q\frac{1}{\mu(Q)}\int_Q|f(x)-f_Q|\,d\mu(x),$$

where $f_Q=\dfrac{1}{\mu(Q)}\displaystyle\int_Q f(x)d\mu(x)$ *and the supremum is taken over all balls* Q *in* X.

Then, the space BMO is defined by

$$BMO=\left\{f\in L^1_{loc}(X):\|f\|_*<\infty\right\}.$$

Below are the definitions of Calderón-Zygmund operators of theta type and their commutators.

Definition 1.4. *Let* θ *be a nonnegative, nondecreasing function on* $(0,\infty)$ *with* $\int_0^1\theta(t)t^{-1}dt<\infty$ *(1.3). Let* $V(x,y)=\mu\left(B(x.d(x,y))\right)$. *Suppose that* $K(x,y)$ *is a continuous function on* $X\times X\setminus\{(x,x):x\in X\}$ *and satisfies the following conditions:*

(i) $$|K(x,y)|\le\frac{C}{V(x,y)}. \qquad (1.4)$$

(ii) $$|K(x,y)-K(x_0,y)|+|K(y,x)-K(y,x_0)|\le\frac{C}{V(x_0,y)}\theta\left(\frac{d(x,x_0)}{d(y,x_0)}\right), \qquad (1.5)$$

for every x,x_0,y *with* $2A_0d(x,x_0)<d(y,x_0)$. *Then,* $K(x,y)$ *is called a standard kernel of type* θ.

Definition 1.5. *Let* θ *be a function as in Definition 1.4 and a linear operator* T *from* $\mathcal{S}(X)$*, the space of Schwartz functions on X, to* $\mathcal{S}'(X)$ *satisfy the following conditions:*

(i) T *is bounded on* $L^2(X)$*, which means*

$$\|Tf\|_{L^2}\le C\|f\|_{L^2},\quad \text{for every } f\in C_0^{\infty}(X). \qquad (1.6)$$

(ii) There exists a standard kernel K *of type* θ *such that for every function* $f\in C_0^{\infty}(X)$ *and* $x\notin supp(f)$

$$Tf(x)=\int_X K(x,y)f(y)dy. \qquad (1.7)$$

Then T *is called a Calderón-Zygmund operator of type* θ.

Definition 1.6. *Let* T *be a Calderón-Zygmund operator of type* θ *and* $b \in L^1_{loc}(X)$. *The commutator* $[b,T]$ *is defined by*

$$[b,T]f(x) := b(x)T(f)(x) - T(bf)(x)$$

for functions $f \in L^1_{loc}(X)$.

Definition 1.7. *Let* $q > 0$. *The modified Hardy – Littlewood maximal function and sharp maximal function are defined respectively by*

$$M_q f(x) = M(|f|^q)^{1/q}(x) = \left(\sup_{r>0} \frac{1}{\mu(B)} \int_B |f(y)|^q \, dy \right)^{1/q},$$

$$M_q^{\#} f(x) = \sup_{r>0} \inf_{c\in\mathbb{R}} \left(\frac{1}{\mu(B)} \int_B \left| |f(y)|^q - |c|^q \right| dy \right)^{1/q},$$

where $B = B(x,r)$ *is the ball of radius* r *centered at x.*

2. Key lemmas

We first derive here a version of the lemma of Calderón-Zygmund decomposition on spaces of homogeneous type.

Lemma 2.1. Let $f \in L^1(X)$ and $\alpha > 0$. Then, there exists a family of disjoint balls $\{B_j\}_{j=1}^{\infty}$ and functions g and $b_j, j = 1, 2 \ldots$ such that

(i) $f = g + \sum_j b_j$;

(ii) $|g(x)| \le C\alpha, \forall x \in X$ and $\|g\|_{L^1} \le C\|f\|_{L^1}$;

(iii) $\operatorname{supp}(b_j) \subset B_j, \int_X b_j(y)\, d\mu(y) = 0$ and $\sum_j \|b_j\|_{L^1} \le C\|f\|_{L^1}$;

(iv) $\sum_j \mu(B_j^*) \le \frac{C}{\alpha}\|f\|_{L^1}$, where $B_j^* = (5A_0^2)B_j$.

Proof: Note that the balls $(5A_0^2)B_j$ satisfy Vitali's covering lemma for spaces of homogeneous type in the well-known proof of Calderón-Zygmund decomposition in $\mathbb{R}^n$, so the proof of Lemma 2.1 can be obtained similarly. We thus omit the details.

Lemma 2.2. If $\operatorname{supp}(b_j) \subset B$ and $\int_X b_j(y)\, d\mu(y) = 0$ then

$$\int_{X\setminus B^*}\left|Tb_j(x)\right|d\mu(x)\le C\left\|b_j\right\|_{L^1}.$$

Proof:

Write $B=B(x,r)$. Let $B^*=(5A_0)B,\ B^{**}=(5A_0)B^*$. Note that for $x\notin B^*$ and $y\in B$, we have

$$2A_0d(x_0,y)\le 2A_0r<5A_0r\le d(x_0,x).$$

Now we can estimate as follows

$$\left|Tb_j(x)\right|=\left|\int_X K(x,y)b_j(y)d\mu(y)\right|\le\int_B\left|K(x,y)-K(x,x_0)\right|\left|b_j(y)\right|d\mu(y)$$
$$\le\int_B\frac{C}{V(x,x_0)}\theta\left(\frac{d(x_0,y)}{d(x_0,x)}\right)\left|b_j(y)\right|d\mu(y).$$

Then we deduce that

$$\int_{X\setminus B^*}|Tb_j(x)|\,d\mu(x)\lesssim\int_{X\setminus B^*}\int_B\frac{1}{V(x,x_0)}\theta\left(\frac{d(x_0,y)}{d(x_0,x)}\right)\left|b_j(y)\right|d\mu(y)d\mu(x)$$
$$=\int_B\left[\sum_{i=1}^{\infty}\int_{5^{i+1}A_0^2B\setminus 5^iA_0^2B}\frac{1}{V(x,x_0)}\theta\left(\frac{d(x_0,y)}{d(x_0,x)}\right)\left|b_j(y)\right|d\mu(x)\right]d\mu(y)$$
$$\lesssim\int_B\left[\sum_{i=1}^{\infty}\theta\left(5^{-i}\right)\left|b_j(y)\right|\right]d\mu(y)\lesssim\int_B\left[\int_0^1\frac{\theta(t)}{t}dt\right]\left|b_j(y)\right|d\mu(y)\lesssim\|b\|_{L^1}.$$

Lemma 2.3. *Let* $1<q<\infty$ and T *be a Calderón-Zygmund operator defined in Definition 1.4. Then* T *is of strong type* (q,q).

Proof: Fix $\alpha>0$. We write $f=g+b$, where $b=\sum_j b_j$. Then $Tf=Tg+Tb$.

Set $\lambda_f(t)=\mu\left(\{x\in X: f(x)\ge t\}\right)$. We have $\lambda_{Tf}(\alpha)\le\lambda_{Tg}\left(\frac{\alpha}{2}\right)+\lambda_{Tb}\left(\frac{\alpha}{2}\right)$.

By Chebyshev's inequality and Lemma 2.1, we obtain

$$\lambda_{Tg}\left(\frac{\alpha}{2}\right)\le\frac{1}{(\alpha/2)^2}\int_{|f|\ge\frac{\alpha}{2}}|Tg(y)|^2\,d\mu(y)\le C\int_X|g(y)|^2\,d\mu(y)\le\frac{C}{\alpha}\int_X\left|g(y)\right|d\mu(y)\le\frac{C}{\alpha}\|f\|_{L^1}.$$

Let $E=\bigcup_j B_j^{**}$. It follows from the condition (iv) of Lemma 2.1 and Chebyshev's inequality that

$$\lambda_{Tb}\left(\frac{\alpha}{2}\right)\le\mu(E)+\mu\left(\left\{x\in E^c:\left|Tb(x)\right|>\frac{\alpha}{2}\right\}\right)\le\frac{C}{\alpha}\|f\|_{L^1}+\frac{2}{\alpha}\|Tb\|_{L^1(E^c)}.$$

On one hand, it is clear that

$$\|Tb\|_{L^1(E^c)} \le \sum_j \|Tb_j\|_{L^1(E^c)} \le \sum_j \|Tb_j\|_{L^1\left(B_j^{**}\right)^c}.$$

Since $\operatorname{supp}(b_j) \subset B_j^*$, applying Lemma 2.1 gives

$$\int_{\left(B_j^{**}\right)^c} |Tb_j(x)| \, d\mu(x) \le C \|b_j\|_{L^1}.$$

Therefore, we deduce that

$$\|Tb\|_{L^1(E^c)} \le C \sum_j \|b_j\|_{L^1} \le C \| f \|_{L^1}.$$

As a result of the estimates above, we derive $\lambda_{Tf}(\alpha) \le \frac{C}{\alpha} \| f \|_{L^1}$, which implies the weak type $(1,1)$ of Tf. By using interpolation and the boundedness of T in L^2, we establish the strong type (p, p) of T, for every $1 < q \le 2$. Finally, by using duality, we get the strong type (q,q) of T, for every $1 < q < \infty$ and hence complete the proof of Lemma 2.3.

By following verbatim the proof of Theorem 2.8 (Kinnunen, 2007), we obtain the Kolmogorov inequality version for spaces of homogeneous type as follows.

Theorem 2.4. (Kolmogorov inequality) *Let $p \in [1,\infty)$ and T be a sublinear operator from $L^p(X)$ to the space of measurable functions on X.*

(i) If T is of weak type (p, p), then for all $q \in (0, p)$ and every set $A \subset X$ such that $0 < \mu(A) < \infty$, there exists a constant $C > 0$ such that

$$\int_A | Tf(x) |^q \, dx \le c\mu(A)^{1-q/p} \| f \|_p^q. \qquad (2.1)$$

(ii) If there exists $r \in (0, p)$ and a constant $C > 0$ such that (2.1) holds for every set A with finite measure and $f \in L^p(X)$, then T is of weak type (p, p).

Lemma 2.5. *Let $p \in (1,\infty)$, $r \in [1,\infty)$ and $\varphi(t)$ satisfy (1.1). For every $0 < q < p$, there exists a positive constant C, which depends only on p,q,r, such that*

$$\|M_q(f)\|_{M_\varphi^{p,r}} \le C \| f \|_{M_\varphi^{p,r}}.$$

Proof: The proof of Lemma 2.5 can be easily obtained by following verbatim the proof of Lemma 2.2 in the paper of Nghia et al. (2023). So, we omit the details.

Lemma 2.6. *Let $q > 1, p \in (1,\infty), r \in [1,\infty]$ and $\varphi(t)$ satisfy (1.1). Suppose that T is a Calderón-Zygmund operator of type θ. Then there exists a constant $C > 0$ such that*

$$M^{\#}(Tf)(x_0) \le CM_q f(x_0), \text{ for every } x_0 \in X.$$

Proof: We first prove the following statement: for any $q>1$, any ball $B= B(x_0,r)$ and $c=c_q$ there exists $C=C_q$ such that

$$\frac{1}{\mu(B)}\int_X \left||Tf(x)|-|c|\right| d\mu(x) \le CM_q f(x_0). \quad (2.2)$$

By writing $f=f_1+f_2$, where $f_1=f\chi_{2A_0B}$ and $f_2=f\chi_{X\setminus 2A_0B}$, and choosing $c=(Tf_2)_B=\frac{1}{\mu(B)}\int_X Tf_2(x)d\mu(x)$, we can estimate the left-hand side of (2.1) as follows:

$$\frac{1}{\mu(B)}\int_B \left||Tf(x)|-|(Tf_2)_B|\right| d\mu(x) \le \frac{1}{\mu(B)}\int_B \left|Tf(x)-(Tf_2)_B\right| d\mu(x)$$

$$=\frac{1}{\mu(B)}\int_B \left|Tf(x)_1+\left(Tf_2(x)-(Tf_2)_B\right)\right| d\mu(x)$$

$$\le \frac{1}{\mu(B)}\int_B |Tf_1(x)|d\mu(x)+\frac{1}{\mu(B)}\int_B \left|Tf_2(x)-(Tf_2)_B\right| d\mu(x).$$

Set $I_1=\frac{1}{\mu(B)}\int_B |Tf_1|d\mu(x)$ and $I_2=\frac{1}{\mu(B)}\int_B \left|Tf_2-(Tf_2)_B\right|d\mu(x)$.

Given Lemma 2.3, we get the strong type (q,q) of T. Thus, T is an operator of weak type (q,q). Then, it follows from the Kolmogorov inequality that

$$\frac{1}{\mu(B)}\int_B |Tf_1|d\mu(x) \lesssim \frac{1}{\mu(B)^{\frac{1}{q}}}\left(\int_X |f_1|^q d\mu(x)\right)^{\frac{1}{q}}.$$

Therefore, we can now estimate I_1 as follows

$$I_1 \lesssim \frac{1}{\mu(B)^{\frac{1}{q}}}\left(\int_X |f_1|^q d\mu(x)\right)^{\frac{1}{q}} = \frac{1}{\mu(B)^{\frac{1}{q}}}\left(\int_{2A_0B} |f|^q d\mu(x)\right)^{\frac{1}{q}}$$

$$\lesssim \frac{1}{\mu(2A_0B)^{\frac{1}{q}}}\left(\int_{2A_0B} |f|^q d\mu(x)\right)^{\frac{1}{q}} \lesssim M_q f(x_0).$$

For I_2, using the definition of Calderón-Zygmund operator T yields

$$I_2=\frac{1}{\mu(B)}\int_B \left|Tf_2-(Tf_2)_B\right|d\mu(x)$$

$$=\frac{1}{\mu(B)}\int_B\left|\int_X K(x,y)\,f_2(y)\,d\mu(y)-\frac{1}{\mu(B)}\int_B\int_X K(z,y)\,f_2(y)\,d\mu(y)d\mu(z)\right|d\mu(x)$$

$$\le\frac{1}{\mu(B)}\int_B\left|\frac{1}{\mu(B)}\int_B\int_X K(x,y)\,f_2(y)\,d\mu(y)\,d\mu(z)-\frac{1}{\mu(B)}\int_B\int_X K(z,y)\,f_2(y)\,d\mu(y)d\mu(z)\right|d\mu(x)$$

$$\le\frac{1}{\left[\mu(B)\right]^2}\int_B\int_B\int_X\left|f_2(y)\right|\left|K(x,y)-K(z,y)\right|d\mu(y)\,d\mu(z)\,d\mu(x)$$

$$\le\frac{1}{\left[\mu(B)\right]^2}\int_B\int_B\int_X\left|f_2(y)\right|\left|K(x,y)-K(x_0,y)\right|d\mu(y)d\mu(z)d\mu(x)$$

$$+\frac{1}{\left[\mu(B)\right]^2}\int_B\int_B\int_X\left|f_2(y)\right|\left|K(z,y)-K(x_0,y)\right|d\mu(y)d\mu(z)d\mu(x).$$

For $z,x\in B$ and $y\notin 2A_0B$, it is clear that $2A_0d(x,x_0)<d(y,x_0)$ and $2A_0d(z,x_0)<d(y,x_0)$. In light of the regularity condition of kernel K, we have.

$$\int_X\left|f_2(y)\right|\left|K(x,y)-K(x_0,y)\right|d\mu(y)=\int_{X\setminus 2A_0B}\left|f(y)\right|\left|K(x,y)-K(x_0,y)\right|d\mu(y)$$

$$\lesssim\sum_{j=1}^{\infty}\int_{2^{j+1}A_0B\setminus 2^jA_0B}\frac{\theta\left(d(x,x_0)/d(y,x_0)\right)}{V(x_0,y)}\left|f(y)\right|d\mu(y)\lesssim\sum_{j=1}^{\infty}\theta\left(2^{-j}\right)\frac{1}{\mu\left(2^{j+1}A_0B\right)}\left(\int_{2^{j+1}A_0B}\left|f^q(y)\right|d\mu(y)\right)^{\frac{1}{q}}$$

$$\lesssim\sum_{j=1}^{\infty}\theta\left(2^{-j}\right)M_qf(x_0)\lesssim M_qf(x_0)\int_0^1\frac{\theta(t)}{t}dt\lesssim M_qf(x_0).$$

Similarly, we have

$$\int_X\left|f_2(y)\right|\left|K(z,y)-K(x_0,y)\right|d\mu(y)\lesssim M_qf(x_0).$$

Therefore, the following estimate holds true.

$$I_2\lesssim\frac{1}{\mu(B)}\cdot\frac{1}{\mu(B)}\int_B\int_B M_qf(x_0)\,d\mu(z)d\mu(x)\lesssim M_qf(x_0).$$

Finally, it follows from the estimates above that

$$\frac{1}{\mu(B)}\int_B\left|\left|Tf\right|-\left|c\right|\right|d\mu\lesssim I_1+I_2\lesssim M_qf(x_0).$$

That means

$$M^{\#}(Tf)(x_0)\lesssim M_qf(x_0).$$

Lemma 2.7. Let $B_1=B(x_1,r_1)$, $B_2=B(x_2,r_2)$ be balls whose intersection is not empty and $r_2\le A_0r_1\le 2A_0r_2$. Then we have

$$\left|b_{B_1}-b_{B_2}\right|\le 2C\|b\|_{BMO}, \forall b\in BMO(X).$$

Proof: Take $x\in B_1\cap B_2$ and let $B=B\left(x_1, 2(1+A_0+A_0^2)r_2\right)$. We can easily verify that $B_1\subset B, B_2\subset B$ and $\mu(B)\lesssim \mu(B_1), \mu(B)\lesssim\mu(B_2)$.

Then

$$\left|b_{B_1}-b_B\right|=\left|\frac{1}{\mu(B_1)}\int_{B_1} b_B d\mu(y)-\frac{1}{\mu(B_1)}\int_{B_1} b(y)d\mu(y)\right|\le\frac{C}{\mu(B_1)}\int_{B_1}\left|b(y)-b_B\right|d\mu(y)\le C\|b\|_{BMO}.$$

By an argument analogous to $\left|b_B-b_{B_1}\right|$, we also obtain $\left|b_B-b_{B_2}\right|\le C\|b\|_{BMO}$.

Hence, we derive

$$\left|b_{B_1}-b_{B_2}\right|\le\left|b_{B_1}-b_B\right|+\left|b_B-b_{B_2}\right|\le 2C\|b\|_{BMO}.$$

Lemma 2.8. Let $b\in BMO(X)$ and T be a Calderón-Zygmund operator of type θ with $\int_0^1\theta(t)t^{-1}\left|\log t\right|dt<\infty$. Then, for any $q\in(1,p)$, there exists a positive constant C such that

$$M^{\#}\left([b,T](f)\right)(x_0)\le C\|b\|_{BMO}\left(M_q\left(T(f)\right)(x_0)+M_q f(x_0)\right)$$

for any $f\in M_\varphi^{p,r}(X)$ and $x_0\in X$.

Proof: First, we prove for each $q>1$, each ball $B(x_0,r)$ and for some constant $c=c_q$, there exists $C=C_q>0$ such that

$$\left(\frac{1}{\mu(B)}\int_B\left|\left|[b,T]f(y)\right|-|c|\right|d\mu(y)\right)\le C\|b\|_{BMO}\left(M_q\left(T(f)\right)(x_0)+M_q f(x_0)\right).$$

We write $f=f_1+f_2$, with $f_1=f\chi_{(2A_0)B}$, $f_2=f\chi_{X\setminus(2A_0)B}$ and let

$$c=-\left(T\left(\left(b-b_{(2A_0)B}\right)f_2\right)\right)_B.$$

It is easy to check that

$$[b,T]f=b.T(f)-T(bf)=\left(b-b_{(2A_0)B}\right)Tf-T\left(\left(b-b_{(2A_0)B}\right)f_1\right)-T\left(\left(b-b_{(2A_0)B}\right)f_2\right).$$

Then

$$\frac{1}{\mu(B)}\int_B\left|\left|[b,T]f(y)\right|-|c|\right|d\mu(y)\le\frac{1}{\mu(B)}\int_B\left|[b,T]f(y)+\left(T\left(\left(b-b_{(2A_0)B}\right)f_2\right)\right)_B\right|d\mu(y)$$

$$\lesssim \frac{1}{\mu(B)}\int_B \left|\left(b(y)-b_{(2A_0)B}\right)\right|\left|Tf(y)\right| d\mu(y) + \frac{1}{\mu(B)}\int_B \left|T\left(\left(b-b_{(2A_0)B}\right)f_1\right)(y)\right| d\mu(y)$$

$$+\frac{1}{\mu(B)}\int_B \left|T\left(\left(b-b_{(2A_0)B}\right)f_2\right)(y)-\left(T\left(\left(b-b_{(2A_0)B}\right)f_2\right)\right)_B\right| d\mu(y).$$

Set

$$I_1 = \frac{1}{\mu(B)}\int_B \left|\left(b(y)-b_{(2A_0)B}\right)\right|\left|Tf(y)\right| d\mu(y),$$

$$I_2 = \frac{1}{\mu(B)}\int_B \left|T\left(\left(b-b_{(2A_0)B}\right)f_1\right)(y)\right| d\mu(y),$$

$$I_3 = \frac{1}{\mu(B)}\int_B \left|T\left(\left(b-b_{(2A_0)B}\right)f_2\right)(y)-\left(T\left(\left(b-b_{(2A_0)B}\right)f_2\right)\right)_B\right| d\mu(y).$$

For I_1, by using Holder's inequality with exponents q and q' satisfying $\frac{1}{q}+\frac{1}{q'}=1$, we get

$$I_1 \lesssim \left(\frac{1}{\mu(B)}\int_B \left|\left(b(y)-b_{(2A_0)B}\right)\right|^{q'} d\mu(y)\right)^{\frac{1}{q'}}\left(\frac{1}{\mu(B)}\int_B \left|Tf(y)\right|^q d\mu(y)\right)^{\frac{1}{q}} \lesssim \|b\|_{BMO} M_q\left(T(f)\right)(x_0)$$

For I_2, take any $u\in(1,q)$. Since T is of weak type (u,u), we can apply Kolmogorov's inequality to get

$$I_2 \le \frac{1}{\mu(B)^{\frac{1}{u}}}\left(\int_X \left|\left(b(y)-b_{(2A_0)B}\right)f_1(y)\right|^u d\mu(y)\right)^{\frac{1}{u}}$$

$$\lesssim \frac{1}{\mu\left((2A_0)B\right)^{\frac{1}{u}}}\left(\int_{(2A_0)B} \left|\left(b(y)-b_{(2A_0)B}\right)f(y)\right|^u d\mu(y)\right)^{\frac{1}{u}}$$

$$\lesssim \left(\frac{1}{\mu\left((2A_0)B\right)}\int_{(2A_0)B} \left|\left(b(y)-b_{(2A_0)B}\right)\right|^{\frac{qu}{q-u}} d\mu(y)\right)^{\frac{qu}{q-u}}\left(\frac{1}{\mu\left((2A_0)B\right)}\int_{(2A_0)B} |f(y)|^q d\mu(y)\right)^{\frac{1}{q}}$$

$$\lesssim \|b\|_{BMO} M_q(f)(x_0).$$

For I_3, it is clear to see that

$$I_3 = \frac{1}{\mu(B)}\int_B \left|T\left(\left(b-b_{(2A_0)B}\right)f_2\right)(y)-\left(T\left(\left(b-b_{(2A_0)B}\right)f_2\right)\right)_B\right| d\mu(y)$$

$$=\frac{1}{\mu(B)}\int_B \left|\int_X K(y,w).\left(b(w)-b_{2A_0B}\right)f_2(w)d\mu(w)-\frac{1}{\mu(B)}\int_B\int_X K(z,w)\left(b(w)-b_{2A_0B}\right)f_2(w)d\mu(w)d\mu(z)\right| d\mu(y)$$

$$\leq \frac{1}{\mu(B)^2}\int_B\int_B\int_{X\setminus 2A_0B}\left|K(y,w)-K(z,w)\right|.\left|b(w)-b_{2A_0B}\right|.\left|f(w)\right|d\mu(w)d\mu(z)d\mu(y)$$

$$\lesssim \frac{1}{\mu(B)^2}\int_B\int_B\sum_{j=1}^{\infty}\int_{2^{j+1}A_0B\setminus 2^jA_0B}\left|K(y,w)-K(z,w)\right|\left|b(w)-b_{2A_0B}\right|.\left|f(w)\right|d\mu(w)d\mu(z)d\mu(y)$$

$$\lesssim \frac{1}{\mu(B)^2}\int_B\int_B\sum_{j=1}^{\infty}\int_{2^{j+1}A_0B\setminus 2^jA_0B}\frac{\theta\left((2A_0)^{-j}\right)}{\mu\left(2^{j+1}A_0B\right)}.\left|b(w)-b_{2A_0B}\right|.\left|f(w)\right|d\mu(w)d\mu(z)d\mu(y)$$

$$\lesssim \sum_{j=1}^{\infty}\theta\left((2A_0)^{-j}\right)\int_{2^{j+1}A_0B}\frac{1}{\mu\left(2^{j+1}A_0B\right)}\left|b(w)-b_{2A_0B}\right|.\left|f(w)\right|d\mu(w)$$

$$\lesssim \sum_{j=1}^{\infty}\theta\left((2A_0)^{-j}\right)\int_{2^{j+1}A_0B}\frac{1}{\mu\left(2^{j+1}A_0B\right)}\left|b(w)-b_{2^{j+1}A_0B}\right|.\left|f(w)\right|d\mu(w)$$

$$+\sum_{j=1}^{\infty}\theta\left((2A_0)^{-j}\right)\int_{2^{j+1}A_0B}\frac{1}{\mu\left(2^{j+1}A_0B\right)}\left|b_{2A_0B}-b_{2^{j+1}A_0B}\right|.\left|f(w)\right|d\mu(w).$$

At this stage, we set

$$I_{3a}=\sum_{j=1}^{\infty}\theta\left((2A_0)^{-j}\right)\int_{2^{j+1}A_0B}\frac{1}{\mu\left(2^{j+1}A_0B\right)}\left|b(w)-b_{2^{j+1}A_0B}\right|.\left|f(w)\right|d\mu(w),$$

$$I_{3b}=\sum_{j=1}^{\infty}\theta\left((2A_0)^{-j}\right)\int_{2^{j+1}A_0B}\frac{1}{\mu\left(2^{j+1}A_0B\right)}\left|b_{2A_0B}-b_{2^{j+1}A_0B}\right|.\left|f(w)\right|d\mu(w).$$

According to Holder's inequality with exponents q and q', we have

$$I_{3a}\lesssim\sum_{j=1}^{\infty}\theta\left((2A_0)^{-j}\right)\left(\int_{2^{j+1}A_0B}\frac{1}{\mu\left(2^{j+1}A_0B\right)}\left|b(w)-b_{2^{j+1}A_0B}\right|^{q'}d\mu(w)\right)^{\frac{1}{q'}}.\left(\int_{2^{j+1}A_0B}|f(w|^q d\mu(w)\right)^{\frac{1}{q}}$$

$$\lesssim \|b\|_{BMO}M_q(f)(x_0).$$

It is obvious that $2^kA_0\leq A_02^{k+1}A_0\leq 2A_0.2^kA_0$ with $k=1,2,\ldots,j$. Thus, applying Lemma 2.7 for every pair of balls $2^{k+1}A_0B$ and 2^kA_0B and Holder's inequality implies

$$I_{3b}\lesssim\sum_{j=1}^{\infty}j\theta\left((2A_0)^{-j}\right)\|b\|_{BMO}\frac{1}{\mu\left(2^{j+1}A_0B\right)}\int_{2^{j+1}A_0B}\left|f(w)\right|d\mu(w)$$

$$\lesssim\sum_{j=1}^{\infty}j\theta\left((2A_0)^{-j}\right)\|b\|_{BMO}\left(\frac{1}{\mu(2^{j+1}A_0B)}\int_{2^{j+1}A_0B}|f(w)|^q d\mu(w)\right)^{\frac{1}{q}}$$

$$\lesssim\|b\|_{BMO}M_q(f)(x_0)\int_0^1\frac{\theta(t)}{t}\left|\log t\right|dt\lesssim\|b\|_{BMO}M_q(f)(x_0).$$

Now, combining the estimates for I_{3a}, I_{3b} above together yields

$$I_3 \lesssim \|b\|_{BMO} M_q(f)(x_0).$$

Consequently, there exists $c > 0$ such that

$$\frac{1}{\mu(B)}\int_B \left\| [b,T] f(y) \right| - |c| \, d\mu(y) \lesssim I_1 + I_2 + I_3 \lesssim \|b\|_{BMO}\left(M_q\left(T(f)\right)(x_0) + M_q f(x_0)\right),$$

which gives

$$M^{\#}\left([b,T](f)\right)(x_0) \leq C\|b\|_{BMO}\left(M_q\left(T(f)\right)(x_0) + M_q f(x_0)\right).$$

3. Main result

Theorem 3.1. *Let $p \in (1;\infty)$, $r \in [1;\infty)$, φ satisfy (1.1), T be a Calderón-Zygmund operator of type θ and $b \in BMO(X)$. Then we have the following statements.*

(i) T is bounded on $M_\varphi^{p,r}(X)$ and $\|T(f)\|_{M_\varphi^{p,r}} \lesssim \|f\|_{M_\varphi^{p,r}}, \forall f \in M_\varphi^{p,r}(X)$.

(ii) If θ satisfies the additional condition $\int_0^1 \theta(t) t^{-1} |\log t| dt < \infty$ then

$$\|[b,T](f)\|_{M_\varphi^{p,r}} \lesssim \|b\|_{BMO} \|f\|_{M_\varphi^{p,r}}, \forall f \in M_\varphi^{p,r}(X).$$

Proof: (i) Let $q \in (1, p)$. According to Lemma 2.5, we have $\|M_q(f)\|_{M_\varphi^{p,r}} \lesssim \|f\|_{M_\varphi^{p,r}}$.

Moreover, thanks to Lemma 2.6, it is clear to see that $\|M^{\#}(Tf)\|_{M_\varphi^{p,r}} \lesssim \|M_q(f)\|_{M_\varphi^{p,r}}$.

So, we obtain

$$\|T(f)\|_{M_\varphi^{p,r}} \lesssim \|M^{\#}(Tf)\|_{M_\varphi^{p,r}} \lesssim \|M_q(f)\|_{M_\varphi^{p,r}} \lesssim \|f\|_{M_\varphi^{p,r}}.$$

That is, $T : M_\varphi^{p,r}(X) \to M_\varphi^{p,r}(X)$.

(ii) Let $q \in (1, p)$. By using Lemma 2.5, Lemma 2.6 and (i), we obtain

$$\|[b,T](f)\|_{M_\varphi^{p,r}} \lesssim \|M^{\#}([b,T](f))\|_{M_\varphi^{p,r}} \lesssim \|b\|_{BMO}\left(\|M_q(T(f))\|_{M_\varphi^{p,r}} + \|M_q(f)\|_{M_\varphi^{p,r}}\right)$$

$$\lesssim \|b\|_{BMO}\left(\|T(f)\|_{M_\varphi^{p,r}} + \|f\|_{M_\varphi^{p,r}}\right) \lesssim \|b\|_{BMO}\|f\|_{M_\varphi^{p,r}},$$

which completes the proof of Theorem 3.1.

4. Conclusion

We have extended the main results of Nghia et al. (2023) about the boundedness of Calderón-Zygmund operators and commutators. More specifically, we proved in our main theorem that Calderón-Zygmund operators and commutators of type theta are bounded on Morrey – Lorentz spaces $M_\varphi^{p,r}(X)$, where *X* is a space of homogeneous type.

Theorem 3.1. *Let* $p \in (1;\infty)$, $r \in [1;\infty)$, φ *satisfy (1.1), T be a Calderón-Zygmund operator of type* θ *and* $b \in BMO(X)$. *Then we have the following statements.*

(i) T *is bounded on* $M_{\varphi}^{p,r}(X)$ *and* $\|T(f)\|_{M_{\varphi}^{p,r}} \lesssim \|f\|_{M_{\varphi}^{p,r}}, \forall f \in M_{\varphi}^{p,r}(X)$.

(ii) If θ *satisfies the additional condition* $\int_0^1 \theta(t)t^{-1}|\log t|\,dt < \infty$ *then*

$$\|[b,T](f)\|_{M_{\varphi}^{p,r}} \lesssim \|b\|_{BMO}\|f\|_{M_{\varphi}^{p,r}}, \forall f \in M_{\varphi}^{p,r}(X).$$

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TÍNH BỊ CHẶN CỦA HOÁN TỬ CALDERÓN-ZYGMUND LOẠI THETA TRÊN CÁC KHÔNG GIAN THUẦN NHẤT

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TÓM TẮT

Trong bài báo này, chúng tôi nghiên cứu tính bị chặn của toán tử T là toán tử *Calderón-Zygmund loại* θ *và hoán tử* $[b,T]$ *tương ứng trên các không gian Morrey – Lorentz tổng quát* $M_{\varphi}^{p,r}(X)$, *trong đó* X *là không gian thuần nhất. Đầu tiên, chúng tôi thiết lập bổ đề về sự phân tích Calderón- Zygmund cho không gian* X *loại thuần nhất. Bổ đề này dùng để chứng minh toán tử* T *là toán tử loại mạnh* (p,p) *trên không gian* $L^p(X)$, *với* $p\in(1,\infty)$ *(xem các Bổ đề 2.1, Bổ đề 2.2, Bổ đề 2.3). Sau đó, sử dụng bất đẳng thức Kolmogorov, các điều kiện liên quan đến toán tử* T *và không gian* BMO, *chúng tôi xây dựng hai đánh giá điểm liên quan đến toán tử cực đại nhọn (xem Bổ đề 2.6 và Bổ đề 2.8). Cuối cùng, chúng tôi chứng minh tính bị chặn của* T *và* $[b,T]$ *trên không gian* $M_{\varphi}^{p,r}(X)$ *(xem Định lí 3.1).*

Từ khóa: hoán tử Calderón-Zygmund loại θ; không gian Morrey-Lorentz tổng quát; không gian thuần nhất